

# TOPOLOGY OF THE IMMEDIATE SNAPSHOT COMPLEXES

DMITRY N. KOZLOV

**ABSTRACT.** The immediate snapshot complexes were introduced as combinatorial models for the protocol complexes in the context of theoretical distributed computing. In the previous work we have developed a formal language of witness structures in order to define and to analyze these complexes.

In this paper, we study topology of immediate snapshot complexes. It is known that these complexes are always pure and that they are pseudomanifolds. Here we prove two further independent topological properties. First, we show that immediate snapshot complexes are collapsible. Second, we show that these complexes are homeomorphic to closed balls. Specifically, given any immediate snapshot complex  $P(\bar{r})$ , we show that there exists a homeomorphism  $\varphi : \Delta^{|\text{supp } \bar{r}|-1} \rightarrow P(\bar{r})$ , such that  $\varphi(\sigma)$  is a subcomplex of  $P(\bar{r})$ , whenever  $\sigma$  is a simplex in the simplicial complex  $\Delta^{|\text{supp } \bar{r}|-1}$ .

## 1. WITNESS STRUCTURES AND IMMEDIATE SNAPSHOT PROTOCOL COMPLEXES

### 1.1. Modeling protocol complexes for the immediate snapshot read/write distributed protocols.

A crucial ingredient in the topological approach to theoretical distributed computing, see Herlihy et al, [HKR], is associating a simplicial complex, called the *protocol complex*, to every distributed protocol, once the computational model is fixed. In this paper, we study topology of standard full-information protocol complexes in one of the central models of computation.

Let us fix the computational model to be the immediate snapshot read/write model, which was originally introduced by Borowsky and Gafni in [BG]. Roughly, this means that the processes can write their values to the assigned memory registers, and they can read the entire memory in one atomic step (snapshot read). The execution of the protocol must have a layer structure, where in each layer a group of processes becomes active, the processes in this group atomically write their values to the memory, after this they atomically read the entire memory. Importantly, there are no further restrictions on how these layers get activated during the protocol execution.

In our previous work, [Ko14b], we introduced combinatorial models for the protocol complexes for the standard protocols in that chosen computational model, called *immediate snapshot complexes*. For this, we needed to define new combinatorial structures, called *witness structures*, and study their structure theory, including various operations, such as *ghosting*. We have proved that the immediate snapshot complexes provide the correct model for these protocol complexes, and started to study their topology.

The standard protocols are naturally enumerated by finite sequences of nonnegative integers, which we called *round counters*, denoted  $\bar{r}$ . Accordingly, the immediate snapshot complexes themselves were denoted  $P(\bar{r})$ . In [Ko14b] it was proved that the complexes

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$P(\bar{r})$  are always pseudomanifolds with boundary, and the combinatorics of the boundary subcomplex was described.

In this paper, we improve our understanding of topology of  $P(\bar{r})$  significantly. We refine the notion of canonical subcomplex decomposition of  $P(\bar{r})$  from [Ko14b], and give a complete combinatorial description of the incidence relations in this stratification. This gives us a good approach to understanding the inner structure of  $P(\bar{r})$ . In particular, it is straightforward to prove the contractibility of  $P(\bar{r})$  by pairing the combinatorial description of this incidence structure with the standard result in combinatorial topology, called the *Nerve Lemma*, see [Ko07]. As a first topological property we show a stronger result: namely, that the complexes  $P(\bar{r})$  are always collapsible. The collapsing sequence is also explicitly described.

It takes much more effort to derive the second topological property of  $P(\bar{r})$ , namely the fact that each such complex is homeomorphic to a closed ball of dimension  $|\text{supp } \bar{r}| - 1$ . This is the content of the Corollary 3.15, which is an immediate consequence of our main Theorem 3.14. Specifically, we prove that, for every  $P(\bar{r})$ , there exists a homeomorphism  $\varphi : \Delta^{|\text{supp } \bar{r}| - 1} \rightarrow P(\bar{r})$ , such that  $\varphi(\sigma)$  is a subcomplex of  $P(\bar{r})$ , whenever  $\sigma$  is a simplex in the simplicial complex  $\Delta^{|\text{supp } \bar{r}| - 1}$ .

The work presented here is the rigorous workout of the second part of the preprint [Ko14a]. The detailed expansion of the first part of [Ko14a] has already appeared in [Ko14b], where we laid the combinatorial groundwork for the topological results of this paper. We spend the rest of this section reminding the notations of [Ko14b] and results proved there. Our presentation here is quite condensed and the reader is referred to [Ko14b] for further details. We remark that topology of protocol complexes for related computational models has been studied by many authors, see e.g., [Ha04, HKR, HS, Ko12, Ko13]. Furthermore, we recommend Attiya and Welch, [AW], for an in-depth background on theoretical distributed computing.

Fundamentally, this paper can be viewed a stand-alone article, written in a rigorous mathematical fashion, making it possible, in principle, to be read independently. However, we strongly recommend that the reader consults [Ko14b], before starting reading this one. Furthermore, in order both to facilitate researchers who are mainly interested in distributed computing, as well as topologists interested in more distributed computing background, we shall comment throughout the text, explaining the distributed computing intuition behind the mathematical concepts.

## 1.2. Round counters.

To start with, we review some of the standard terminology which we will use. We let  $\mathbb{Z}_+$  denote the set of nonnegative integers  $\{0, 1, 2, \dots\}$ . For a natural number  $n$  we shall use  $[n]$  to denote the set  $\{0, \dots, n\}$ , with a convention that  $[-1] = \emptyset$ . For a finite subset  $S \subset \mathbb{Z}_+$ , such that  $|S| \geq 2$ , we let  $\text{smax } S$  denote the *second* largest element, i.e.,  $\text{smax } S := \max(S \setminus \{\max S\})$ . Finally, for a set  $S$  and an element  $a$ , we set

$$\chi(a, S) := \begin{cases} 1, & \text{if } a \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, whenever  $(X_i)_{i=1}^t$  is a family of topological spaces, we set  $X_I := \bigcap_{i \in I} X_i$ . Also, when no confusion arises, we identify one-element sets with that element, and write, e.g.,  $p$  instead of  $\{p\}$ .

Next, we proceed to the combinatorial enumeration of all standard protocols, together with relation terminology. This is accomplished by the introduction of the so-called *round counters*.

**Definition 1.1.** Given a function  $\bar{r} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\perp\}$ , we consider the set

$$\text{supp } \bar{r} := \{i \in \mathbb{Z}_+ \mid \bar{r}(i) \neq \perp\}.$$

This set is called the **support set** of  $\bar{r}$ .

A **round counter** is a function  $\bar{r} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\perp\}$  with a finite support set.

Obviously, a round counter can be thought of as an infinite sequence  $\bar{r} = (\bar{r}(0), \bar{r}(1), \dots)$ , where, for all  $i \in \mathbb{Z}_+$ , either  $\bar{r}(i)$  is a nonnegative integer, or  $\bar{r}(i) = \perp$ , such that only finitely many entries of  $\bar{r}$  are nonnegative integers. We shall frequently use a short-hand notation  $\bar{r} = (r_0, \dots, r_n)$  to denote the round counter given by

$$\bar{r}(i) = \begin{cases} r_i, & \text{for } 0 \leq i \leq n; \\ \perp, & \text{for } i > n. \end{cases}$$

**Definition 1.2.** Given a round counter  $\bar{r}$ , the number  $\sum_{i \in \text{supp } \bar{r}} \bar{r}(i)$  is called the **cardinality** of  $\bar{r}$ , and is denoted  $|\bar{r}|$ . The sets

$$\text{act } \bar{r} := \{i \in \text{supp } \bar{r} \mid \bar{r}(i) \geq 1\} \text{ and } \text{pass } \bar{r} := \{i \in \text{supp } \bar{r} \mid \bar{r}(i) = 0\}$$

are called the **active** and the **passive** sets of  $\bar{r}$ .

**Distributed Computing Context 1.3.** Since we consider full-information protocols only, they can be described by specifying the number of rounds each process executes the write-read sequence. Mathematically, these protocols are indexed by round counters. Given a round counter  $\bar{r}$ , the set  $\text{supp } \bar{r}$  indexes the participating processes, and is required to be finite. The symbol  $\perp$  means that the process does not participate. Accordingly, the set  $\text{pass } \bar{r}$  indexes the passive processes, i.e., those, which formally take part in the execution, but which do not actually perform any active steps, while the set  $\text{act } \bar{r}$  indexes the processes which execute at least one step.

The following special class of round counters is important for our study.

**Definition 1.4.** For an arbitrary pair of disjoint finite sets  $A, B \subseteq \mathbb{Z}_+$  we define a round counter  $\chi_{A,B}$  given by

$$\chi_{A,B}(i) := \begin{cases} 1, & \text{if } i \in A; \\ 0, & \text{if } i \in B. \end{cases}$$

Furthermore, for an arbitrary round counter  $\bar{r}$ , we set  $\chi(\bar{r}) := \chi_{\text{act } \bar{r}, \text{pass } \bar{r}}$ .

We note that  $\text{supp } \bar{r} = \text{supp } (\chi(\bar{r}))$ . In the paper we shall also use the short-hand notation  $\chi_A := \chi_{A, \emptyset}$ .

We define two operations on the round counters. To start with, assume  $\bar{r}$  is a round counter and we have a subset  $A \subseteq \mathbb{Z}_+$ . We let  $\bar{r} \setminus A$  denote the round counter defined by

$$(\bar{r} \setminus A)(i) = \begin{cases} \bar{r}(i), & \text{if } i \notin A; \\ \perp, & \text{if } i \in A. \end{cases}$$

We say that the round counter  $\bar{r} \setminus A$  is obtained from  $\bar{r}$  by the *deletion* of  $A$ . Note that  $\text{supp } (\bar{r} \setminus A) = \text{supp } (\bar{r}) \setminus A$ ,  $\text{act } (\bar{r} \setminus A) = \text{act } (\bar{r}) \setminus A$ , and  $\text{pass } (\bar{r} \setminus A) = \text{pass } (\bar{r}) \cup A$ .

Furthermore, we have  $\chi(\bar{r} \setminus A) = \chi(\bar{r}) \setminus A$ . Finally, we note for future reference that for  $A \subseteq C \cup D$  we have

$$(1.1) \quad \chi_{C,D} \setminus A = \chi_{C \setminus A, D \setminus A}.$$

For the second operation, assume  $\bar{r}$  is a round counter and we have a subset  $S \subseteq \text{act } \bar{r}$ . We let  $\bar{r} \downarrow S$  denote the round counter defined by

$$(\bar{r} \downarrow S)(i) = \begin{cases} \bar{r}(i), & \text{if } i \notin S; \\ \bar{r}(i) - 1, & \text{if } i \in S. \end{cases}$$

We say that the round counter  $\bar{r} \downarrow S$  is obtained from  $\bar{r}$  by the *execution* of  $S$ . Note that  $\text{supp}(\bar{r} \downarrow S) = \text{supp } \bar{r}$ ,  $\text{act}(\bar{r} \downarrow S) = \{i \in \text{act } \bar{r} \mid i \notin S, \text{ or } \bar{r}(i) \geq 2\}$ , and  $\text{pass}(\bar{r} \downarrow S) = \text{pass}(\bar{r}) \cup \{i \in S \mid \bar{r}(i) = 1\}$ . However, in general we have  $\chi(\bar{r}) \downarrow S \neq \chi(\bar{r} \downarrow S)$ .

**Distributed Computing Context 1.5.** *The replacement of  $\bar{r}$  with  $\bar{r} \setminus A$  yields a new protocol, where all processes from  $A$  have been banned from participation. The replacement of  $\bar{r}$  with  $\bar{r} \downarrow S$  corresponds to letting processes from  $S$  execute one round, and then running the remaining protocol with new inputs.*

For an arbitrary round pointer  $\bar{r}$  and sets  $S \subseteq \text{act } \bar{r}$ ,  $A \subseteq \text{supp } \bar{r}$  we set

$$(1.2) \quad \bar{r}_{S,A} := (\bar{r} \downarrow S) \setminus A = (\bar{r} \setminus A) \downarrow (S \setminus A).$$

In the special case, when  $A \cap S = \emptyset$ , the identity (1.2) specializes to

$$(1.3) \quad \bar{r}_{S,A} := (\bar{r} \downarrow S) \setminus A = (\bar{r} \setminus A) \downarrow S.$$

When  $A = \emptyset$ , we shall frequently use the short-hand notation  $\bar{r}_S$  instead of  $\bar{r}_{S,A}$ , in other words,  $\bar{r}_S = \bar{r} \downarrow S$ . Again, for future reference, we note that for  $S \subseteq C$ , we have

$$(1.4) \quad \chi_{C,D} \downarrow S = \chi_{C \setminus S, D \cup S}.$$

### 1.3. Witness structures and the ghosting operation.

Next, we describe the basic terminology which we will need to define the immediate snapshot complexes.

**Definition 1.6.** A **witness prestructure** is a finite sequence of pairs of finite subsets of  $\mathbb{Z}_+$ , denoted  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , with  $t \geq 0$ , satisfying the following conditions:

- (P1)  $W_i, G_i \subseteq W_0$ , for all  $i = 1, \dots, t$ ;
- (P2)  $G_i \cap G_j = \emptyset$ , for all  $i, j \in [t]$ ,  $i < j$ ;
- (P3)  $G_i \cap W_j = \emptyset$ , for all  $i, j \in [t]$ ,  $i \leq j$ .

A witness prestructure is called **stable** if in addition the following condition is satisfied:

- (S) if  $t \geq 1$ , then  $W_t \neq \emptyset$ .

A **witness structure** is a witness prestructure satisfying the following strengthening of condition (S):

- (W) the subsets  $W_1, \dots, W_t$  are all nonempty.

**Definition 1.7.** We define the following data associated to an arbitrary witness prestructure  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ :

- the set  $W_0 \cup G_0$  is called the **support** of  $\sigma$  and is denoted by  $\text{supp } \sigma$ ;
- the **ghost set** of  $\sigma$  is the set  $G(\sigma) := G_0 \cup \dots \cup G_t$ ;
- the **active set** of  $\sigma$  is the complement of the ghost set

$$A(\sigma) := \text{supp } (\sigma) \setminus G(\sigma) = W_0 \setminus (G_1 \cup \dots \cup G_t);$$

- the **dimension** of  $\sigma$  is

$$\dim \sigma := |A(\sigma)| - 1 = |W_0| - |G_1| - \dots - |G_t| - 1.$$

For brevity of some formulas, we set  $W_{-1} := W_0 \cup G_0 = \text{supp } \sigma$ .

**Definition 1.8.** For a prestructure  $\sigma$  and an arbitrary  $p \in \text{supp } \sigma$ , we set

$$\text{Tr}(p, \sigma) := \{0 \leq i \leq t \mid p \in W_i \cup G_i\},$$

and call it the **trace** of  $p$ . Furthermore, for all  $p \in \text{supp } \sigma$ , we set

$$\text{last}(p, \sigma) := \max\{-1 \leq i \leq t \mid p \in W_i\}.$$

When the choice of  $\sigma$  is unambiguous, we shall simply write  $\text{Tr}(p)$  and  $\text{last}(p)$ . The following definition provides an alternative approach to witness structures using traces.

**Definition 1.9.** A **witness prestructure** is a pair of finite subsets  $A, G \subseteq \mathbb{Z}_+$  together with a family  $\{\text{Tr}(p)\}_{p \in A \cup G}$  of finite subsets of  $\mathbb{Z}_+$ , satisfying the following condition:

(T)  $0 \in \text{Tr}(p)$ , for all  $p \in A \cup G$ .

A witness prestructure is called **stable** if it satisfies an additional condition:

(TS) if  $A = \emptyset$ , then  $\text{Tr}(p) = \{0\}$ , for all  $p \in G$ , else

$$\max_{p \in A} \text{last}(p) \geq \max_{p \in G} \bigcup \text{Tr}(p).$$

Set  $t := \max_{p \in A} \text{last}(p)$ . A stable witness prestructure is called **witness structure** if the following strengthening of Condition (TS) is satisfied:

(TW) for all  $1 \leq k \leq t$  either there exists  $p \in A$  such that  $k \in \text{Tr}(p)$ , or there exists  $p \in G$  such that  $k \in \text{Tr}(p) \setminus \max \text{Tr}(p)$ .

We shall call the form of the presentation of the witness prestructure as a triple  $(A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$  its *trace form*.

**Distributed Computing Context 1.10.** The witness structure is a mathematical object modelling the information which the processes have during the execution of the full-information protocol. Let us explain the distributed computing intuition behind this notation.

The set  $\text{supp } \sigma$  indexes all processes which are participating in the protocol. The processes indexed by the set  $W_0$  are of two different types. Those, whose view is included in  $\sigma$ , and those, who have only been passively witnessed by others. The processes of the first type are indexed by the set  $A(\sigma)$ , the other ones are indexed by the union  $G_1 \cup \dots \cup G_t$ . The set  $G_0$  indexes those processes from  $\text{supp } \sigma$  which have not been witnessed by anybody in this particular execution.

The fact, that  $p \in W_k$  is to be interpreted as “the active participation of process  $p$  in round  $k$  has been witnessed”. This can happen in two ways, either  $p$  itself is active in this execution, or  $p$  is being passively witnessed and this is not the last occurrence of  $p$ . The fact that  $p \in G_k$  means that process  $p$  has been passively witnessed and this is the last occurrence of  $p$ .

We refer the reader to [Ko14b, Section 6], where connection between witness structures and witness posets is explained.

Next, we proceed to describe various operations in witness structures and prestructures. To start with, any stable witness prestructure can be turned into a witness structure, which is called its *canonical form*.

**Definition 1.11.** Assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  is an arbitrary stable witness prestructure. Set  $q := |\{1 \leq i \leq t \mid W_i \neq \emptyset\}|$ . Pick  $0 = i_0 < i_1 < \dots < i_q = t$ , such that

$\{i_1, \dots, i_q\} = \{1 \leq i \leq t \mid W_i \neq \emptyset\}$ . We define the witness structure  $C(\sigma) = ((W_0, G_0), (\tilde{W}_1, \tilde{G}_1), \dots, (\tilde{W}_q, \tilde{G}_q))$ , which is called the **canonical form** of  $\sigma$ , by setting

$$(1.5) \quad \tilde{W}_k := W_{i_k}, \quad \tilde{G}_k := G_{i_{k-1}+1} \cup \dots \cup G_{i_k}, \text{ for all } k = 1, \dots, q,$$

Furthermore, any witness prestructure can be made stable using the following operation.

**Definition 1.12.** Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be a witness prestructure, set

$$q := \max\{0 \leq i \leq t \mid W_i \not\subseteq G(\sigma)\}.$$

The **stabilization** of  $\sigma$  is the witness prestructure  $\text{st}(\sigma)$  whose trace form is  $(A(\sigma), G(\sigma), \{\text{Tr}(p)|_{[q]}\}_{p \in \text{supp } \sigma})$ .

More generally, assume  $S \subseteq A(\sigma)$ , and set

$$q := \max\{0 \leq i \leq t \mid W_i \not\subseteq S \cup G(\sigma)\}.$$

The **stabilization of  $\sigma$  modulo  $S$**  is the witness prestructure  $\text{st}_S(\sigma)$  whose trace form is  $(A(\sigma) \setminus S, G(\sigma) \cup S, \{\text{Tr}(p)|_{[q]}\}_{p \in \text{supp } \sigma})$ .

Combining stabilization modulo a set with taking the canonical form yields a new operation, called *ghosting*, which will be of utter importance for the combinatorial description of the incidence structure in the immediate snapshot complexes.

**Definition 1.13.** For an arbitrary witness structure  $\sigma$ , and an arbitrary  $S \subseteq A(\sigma)$ , we define  $\Gamma_S(\sigma) := C(\text{st}_S(\sigma))$ . We say that  $\Gamma_S(\sigma)$  is obtained from  $\sigma$  by **ghosting  $S$** .

**Distributed Computing Context 1.14.** The operation of ghosting the set of processes  $S$  corresponds to excluding their views from the knowledge that the witness structure encodes. Clearly, the occurrences of processes from  $S$  will not vanish from the witness structure  $\Gamma_S(\sigma)$  altogether, but these processes will cease being active, and whatever we will see of them will just be the residual information passively witnessed by other processes.

The main property of ghosting which one needs for proving the well-definedness of the immediate snapshot complexes is that it behaves well with respect to iterations.

**Proposition 1.15.** Assume  $\sigma$  is a witness structure, and  $S, T \subseteq A(\sigma)$ , such that  $S \cap T = \emptyset$ . Then we have  $\Gamma_T(\Gamma_S(\sigma)) = \Gamma_{S \cup T}(\sigma)$ , expressed functorially we have  $\Gamma_T \circ \Gamma_S = \Gamma_{S \cup T}$ .

#### 1.4. Immediate snapshot complexes.

We have now introduced sufficient terminology in order to describe our main objects of study.

**Definition 1.16.** Assume  $\bar{r}$  is a round counter. We define an abstract simplicial complex  $P(\bar{r})$ , called the **immediate snapshot complex** associated to the round counter  $\bar{r}$ , as follows. The vertices of  $P(\bar{r})$  are indexed by all witness structures  $\sigma = (\{p\}, G, \{\text{Tr}(q)\}_{q \in \{p\} \cup G})$ , satisfying these three conditions:

- (1)  $\{p\} \cup G = \text{supp } \bar{r}$ ;
- (2)  $|\text{Tr}(p)| = r(p) + 1$ ;
- (3)  $|\text{Tr}(q)| \leq r(q) + 1$ , for all  $q \in G$ .

We say that such a vertex has **color  $p$** . In general, the simplices of  $P(\bar{r})$  are indexed by all witness structures  $\sigma = (A, G, \{\text{Tr}(q)\}_{q \in A \cup G})$ , satisfying:

- (1)  $A \cup G = \text{supp } \bar{r}$ ;
- (2)  $|\text{Tr}(q)| = r(q) + 1$ , for all  $q \in A$ ;
- (3)  $|\text{Tr}(q)| \leq r(q) + 1$ , for all  $q \in G$ .

The empty witness structure  $((\emptyset, \text{supp } \bar{r}))$  indexes the empty simplex of  $P(\bar{r})$ . When convenient, we identify the simplices of  $P(\bar{r})$  with the witness structures which index them.

Let  $\sigma$  be a non-empty witness structure satisfying the conditions above. The set of vertices  $V(\sigma)$  of the simplex  $\sigma$  is given by  $\{\Gamma_{A(\sigma) \setminus \{p\}}(\sigma) \mid p \in A\}$ .

Taking boundaries of simplices in  $P(\bar{r})$  corresponds to ghosting of the witness structures. This is only natural taking into account the intuition from the distributed computing context 1.14.

**Proposition 1.17.** Assume  $\bar{r}$  is the round counter, and assume  $\sigma$  and  $\tau$  are simplices of  $P(\bar{r})$ . Then  $\tau \subseteq \sigma$  if and only if there exists  $S \subseteq A(\sigma)$ , such that  $\tau = \Gamma_S(\sigma)$ .

The first property of the simplicial complexes  $P(\bar{r})$ , which is quite easy to see, is that these complexes are pure of dimension  $|\text{supp } \bar{r}| - 1$ . Furthermore, zero values in the round counter have a simple topological interpretation.

**Proposition 1.18.** ([Ko14b, Proposition 4.4]). Assume  $\bar{r} = (r(0), \dots, r(n))$  and  $\bar{r}(n) = 0$ . Let  $\bar{q}$  denote the truncated round counter  $(r(0), \dots, r(n-1))$ . Consider a cone over  $P(\bar{q})$ , which we denote  $P(\bar{q}) * \{a\}$ , where  $a$  is the apex of the cone. Then we have

$$(1.6) \quad P(\bar{r}) \simeq P(\bar{q}) * \{a\},$$

where  $\simeq$  denotes the simplicial isomorphism.

For brevity, we set  $P_n := P(\underbrace{1, \dots, 1}_{n+1})$ . It turns out that the standard chromatic subdivision of an  $n$ -simplex, see [Ko12], is a special case of the immediate snapshot complex.

**Proposition 1.19.** ([Ko14b, Proposition 4.10]). The immediate snapshot complex  $P_n$  and the standard chromatic subdivision of an  $n$ -simplex  $\chi(\Delta^n)$  are isomorphic as simplicial complexes. Explicitly, the isomorphism can be given by

$$(1.7) \quad \Phi : ((B_1, \dots, B_t)(C_1, \dots, C_t)) \mapsto \begin{array}{|c|c|c|c|c|} \hline W_0 & C_1 & C_2 & \dots & C_t \\ \hline [n] \setminus W_0 & B_1 \setminus C_1 & B_2 \setminus C_2 & \dots & B_t \setminus C_t \\ \hline \end{array},$$

where  $W_0 = B_1 \cup \dots \cup B_t$ .

Recall the following property of pure simplicial complexes, strengthening the usual connectivity.

**Definition 1.20.** Let  $K$  be a pure simplicial complex of dimension  $n$ . Two  $n$ -simplices of  $K$  are said to be **strongly connected** if there is a sequence of  $n$ -simplices so that each pair of consecutive simplices has a common  $(n-1)$ -dimensional face. The complex  $K$  is said to be **strongly connected** if any two  $n$ -simplices of  $K$  are strongly connected.

Clearly, being strongly connected is an equivalence relation on the set of all  $n$ -simplices.

**Proposition 1.21.** ([Ko14b, Proposition 5.6]). For an arbitrary round counter  $\bar{r}$ , the simplicial complex  $P(\bar{r})$  is strongly connected.

The next definition describes a weak simplicial analog of being a manifold.

**Definition 1.22.** We say that a strongly connected pure simplicial complex  $K$  is a **pseudo-manifold** if each  $(n-1)$ -simplex of  $K$  is a face of precisely one or two  $n$ -simplices of  $K$ . The  $(n-1)$ -simplices of  $K$  which are faces of precisely one  $n$ -simplex of  $K$  form a simplicial subcomplex of  $K$ , called the **boundary** of  $K$ , and denoted  $\partial K$ .

It was shown in [Ko14b], that immediate snapshot complexes are always pseudomanifolds with boundary.

**Theorem 1.23.** ([Ko14b, Proposition 5.9]). *For an arbitrary round counter  $\bar{r}$ , the simplicial complex  $P(\bar{r})$  is a pseudomanifold, where  $\partial P(\bar{r})$  is the subcomplex consisting of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , satisfying  $G_0 \neq \emptyset$ .*

## 2. A CANONICAL DECOMPOSITION OF THE IMMEDIATE SNAPSHOT COMPLEXES

### 2.1. Definition and examples.

The canonical decomposition of the immediate snapshot complexes has been introduced in [Ko14b]. In order to better understand the topology of these complexes, we need to generalize that definition and look at finer strata.

**Definition 2.1.** *Assume  $\bar{r}$  is a round counter.*

- For every subset  $S \subseteq \text{act } \bar{r}$ , let  $Z_S$  denote the set of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $S \subseteq G_1$ .
- For every pair of subsets  $A \subseteq S \subseteq \text{act } \bar{r}$ , let  $Y_{S,A}$  denote the set of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $R_1 = S$  and  $A \subseteq G_1$ . Furthermore, set  $X_{S,A} := Y_{S,A} \cup Z_S$ .

We shall also use the following short-hand notation:  $X_S := X_{S,\emptyset}$ . This case has been considered in [Ko14b], where it was shown that  $X_S$  is a simplicial subcomplex of  $P(\bar{r})$  for an arbitrary  $S$ .

**Distributed Computing Context 2.2.** *The subcomplexes  $X_S$  correspond to the subset of executions which start with the processes indexed by the set  $S$  executing simultaneously. This explains, from the point of view of distributed computing, why the protocol complex decomposes into these strata.*

On the other extreme, clearly  $Z_S = X_{S,S}$  for all  $S$ . When  $A \not\subseteq S$ , we shall use the convention  $Y_{S,A} = \emptyset$ . Note, that in general the sets  $Y_{S,A}$  need not be closed under taking boundary.

**Proposition 2.3.** *The sets  $X_{S,A}$  are closed under taking boundary, hence form simplicial subcomplexes of  $P(\bar{r})$ .*

**Proof.** Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be a simplex in  $X_{S,A}$ , and assume  $\tau \subset \sigma$ . By Proposition 1.17 there exists  $T \subseteq A(\sigma)$ , such that  $\tau = \Gamma_T(\sigma)$ . By Proposition 1.15 it is enough to consider the case  $|T| = 1$ , so assume  $T = \{p\}$ , and let  $\tau = ((\tilde{W}_0, \tilde{G}_0), \dots, (\tilde{W}_t, \tilde{G}_t))$ .

By definition of  $X_{S,A}$  we have either  $\sigma \in Z_S$  or  $\sigma \in Y_{S,A}$ . Consider first the case  $\sigma \in Z_S$ , so  $S \subseteq G_1$ . Since  $\tilde{G}_1 \supseteq G_1$ , we have  $\tau \in Z_S$ .

Now, assume  $\sigma \in Y_{S,A}$ . This means  $W_1 \cup G_1 = S$  and  $A \subseteq G_1$ . Again  $\tilde{G}_1 \supseteq G_1$  implies  $A \subseteq \tilde{G}_1$ .  $\square$

In particular,  $X_S$  and  $Z_S$  are simplicial subcomplexes of  $P(\bar{r})$ , for all  $S$ . When we are dealing with several round counters, in order to avoid confusion, we shall add  $\bar{r}$  to the notations, and write  $X_{S,A}(\bar{r})$ ,  $X_S(\bar{r})$ ,  $Y_{S,A}(\bar{r})$ ,  $Z_S(\bar{r})$ . We shall also let  $\alpha_{S,A}(\bar{r})$  denote the inclusion map

$$\alpha_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \hookrightarrow P(\bar{r}).$$

### 2.2. The strata of the canonical decomposition as immediate snapshot complexes.

The first important property of the simplicial complexes  $X_{S,A}$  is that they themselves can be interpreted as immediate snapshot complexes. Here, and in the rest of the paper, we shall use  $\rightsquigarrow$  to denote simplicial isomorphisms.



**Proposition 2.4.** *Assume  $A \subseteq S \subseteq \text{act } \bar{r}$ , then there exists a simplicial isomorphism*

$$\gamma_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \xrightarrow{\sim} P(\bar{r}_{S,A}).$$

**Proof.** We start by considering the case  $A = \emptyset$ . Pick an arbitrary simplex  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  belonging to  $X_S$ . By the construction of  $X_S$ , we either have  $W_1 \cup G_1 = S$ , or  $S \subseteq G_1$ . If  $W_1 \cup G_1 = S$ , then set

$$\gamma_S(\sigma) := \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 & G_2 & \dots & G_t \\ \hline \end{array},$$

else  $S \subseteq G_1$ , in which case we set

$$\gamma_S(\sigma) := \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array}.$$

Reversely, assume  $\tau = ((V_0, H_0), \dots, (V_t, H_t))$  is a simplex of  $P(\bar{r}_S)$ . Note, that in any case, we have  $S \subseteq V_0 \cup H_0$ . If  $V_0 \cap S \neq \emptyset$ , we set

$$\rho_S(\tau) := \begin{array}{|c|c|c|c|c|} \hline V_0 \cup (H_0 \cap S) & V_0 \cap S & V_1 & \dots & V_t \\ \hline H_0 \setminus (H_0 \cap S) & H_0 \cap S & H_1 & \dots & H_t \\ \hline \end{array},$$

else  $S \subseteq H_0$ , and we set

$$\rho_S(\tau) := \begin{array}{|c|c|c|c|c|} \hline V_0 \cup S & V_1 & V_2 & \dots & V_t \\ \hline H_0 \setminus S & H_1 \cup S & H_2 & \dots & H_t \\ \hline \end{array}.$$

It is immediate that  $\gamma_S$  and  $\rho_S$  preserve supports,  $A(-)$ ,  $G(-)$ , and hence also the dimension. Furthermore, we can see what happens with the cardinalities of the traces. For all elements  $p$  which do not belong to  $S$ , the cardinalities of their traces are preserved. For all elements in  $S$ , the map  $\gamma_S$  decreases the cardinality of the trace, whereas, the map  $\rho_S$  increases it. It follows that  $\gamma_S$  and  $\rho_S$  are well-defined as dimension-preserving maps between sets of simplices.

To see that  $\gamma_S$  preserves boundaries, pick a top-dimensional simplex  $\sigma = (W_0, S, W_1, \dots, W_t)$  in  $X_S$  and ghost the set  $T$ . Assume first  $S \not\subseteq T$ . In this case not all elements in  $S$  are ghosted. Assume now that  $S \subseteq T$ . This implies that  $\gamma_S$  is well-defined as a simplicial map. Finally, a direct verification shows that the maps  $\gamma_S$  and  $\rho_S$  are inverses of each other, hence they are simplicial isomorphisms.

Let us now consider the case when  $A$  is arbitrary. The simplicial complex  $X_{S,A}$  is a subcomplex of  $X_S$  consisting of all simplices  $\sigma$  satisfying the additional condition  $A \subseteq G_1$ . The image  $\gamma_S(X_{S,A})$  consists of all  $\tau = ((V_0, H_0), \dots, (V_t, H_t))$  in  $P(\bar{r}_{S,A})$  satisfying  $A \subseteq H_0$ . The map  $\Xi : \gamma_S(X_{S,A}) \rightarrow P(\bar{r}_{S,A})$ , taking  $\tau$  to  $((V_0, H_0 \setminus A), (V_1, H_1), \dots, (V_t, H_t))$ , is obviously a simplicial isomorphism, hence the composition  $\gamma_{S,A} = \Xi \circ \gamma_S : X_{S,A} \rightarrow P(\bar{r}_{S,A})$  is a simplicial isomorphism as well.  $\square$

Note that, in particular,

$$\gamma_{A,A}(\sigma) = \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus A & W_1 & W_2 & \dots & W_t \\ \hline G_0 & G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array}.$$

The statement of Proposition 2.4 for the example  $\bar{r} = (2, 1, 1)$ , is shown on Figure 2.1.

The next proposition is a first of several results, which claim commutativity of a certain diagram. All these results have alternative intuitive meaning. For example, the commutativity of the diagram (2.1) can be interpreted as saying that the relation of the stratum  $X_{S \cup A, A}(\bar{r})$  to  $X_{A, A}(\bar{r})$  is the same as the relation of the stratum  $X_S(\bar{r} \setminus A)$  to  $P(\bar{r} \setminus A)$ .

**Proposition 2.5.** Assume  $\bar{r}$  is an arbitrary round counter, and  $S, A \subset \text{act } \bar{r}$ , such that  $S \cap A = \emptyset$ , then the following diagram commutes

$$(2.1) \quad \begin{array}{ccc} X_{A,A}(\bar{r}) & \xleftarrow{i} & X_{S \cup A, A}(\bar{r}) \\ \gamma_{A,A}(\bar{r}) \downarrow & & \searrow \gamma_{S \cup A, A}(\bar{r}) \\ P(\bar{r} \setminus A) & \xleftarrow{\alpha_S(\bar{r} \setminus A)} & X_S(\bar{r} \setminus A) \xrightarrow{\gamma_S(\bar{r} \setminus A)} P(\bar{r}_{S,A}), \end{array}$$

where  $i$  denotes the strata inclusion map.

**Proof.** To start with, note that  $\bar{r}_{S,A} = (\bar{r} \downarrow S) \setminus A = (\bar{r} \downarrow (S \cup A)) \setminus A$ , so the diagram (2.1) is well-defined. To see that it is commutative, pick an arbitrary  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ . We know that either  $A \subseteq G_1$  and  $W_1 \cup G_1 = S \cup A$ , or  $A \cup S \subseteq G_1$ . On one hand, we have

$$(\gamma_{A,A}(\bar{r}) \circ i)(\sigma) = \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus A & W_1 & W_2 & \dots & W_t \\ \hline G_0 & G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array}.$$

On the other hand, we have

$$\gamma_{S \cup A, A}(\bar{r})(\sigma) = \begin{cases} \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array} & \text{if } A \subseteq G_1, W_1 \cup G_1 = S \cup A; \\ \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus (S \cup A) & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S & G_1 \setminus (S \cup A) & G_2 & \dots & G_t \\ \hline \end{array} & \text{if } A \cup S \subseteq G_1. \end{cases}$$

Applying  $\gamma_S(\bar{r} \setminus A)^{-1}$  we can verify that  $\gamma_{A,A}(\bar{r}) \circ i = \alpha_S(\bar{r} \setminus A) \circ \gamma_S(\bar{r} \setminus A)^{-1} \circ \gamma_{S \cup A, A}(\bar{r})$ .  $\square$

**Corollary 2.6.** For any  $A \subseteq \text{act } \bar{r}$ , we have

$$(2.2) \quad X_{A,A}(\bar{r}) = \bigcup_{\emptyset \neq S \subseteq \text{act } \bar{r} \setminus A} X_{S \cup A, A}(\bar{r}) = \bigcup_{A \subset T \subseteq \text{act } \bar{r}} X_{T,A}(\bar{r}).$$

**Proof.** Since  $P(\bar{r} \setminus A) = \bigcup_{\emptyset \neq S \subseteq \text{act } \bar{r} \setminus A} X_S(\bar{r} \setminus A)$ , the equation (2.2) is an immediate consequence of the commutativity of the diagram (2.1).  $\square$

### 2.3. The incidence structure of the canonical decomposition.

Clearly,  $P(\bar{r}) = \bigcup_S X_S(\bar{r})$ . We describe here the complete combinatorics of intersecting these strata.

**Proposition 2.7.** For all pairs of subsets  $A \subseteq S \subseteq \text{supp } \bar{r}$  and  $B \subseteq T \subseteq \text{supp } \bar{r}$  we have:  $X_{S,A} \subseteq X_{T,B}$  if and only if at least one of the following two conditions is satisfied:

- $S = T$  and  $B \subseteq A$ ,
- $T \subseteq A$ .

We remark that it can actually happen that both conditions in Proposition 2.7 are satisfied. This happens exactly when  $S = T = A$ .

**Proof of Proposition 2.7.** First we show that  $T \subseteq A$  implies  $X_{S,A} \subseteq X_{T,B}$ . Take  $\sigma \in X_{S,A}$ . If  $\sigma \in Z_S$ , then we have the following chain of implications:  $S \subseteq G_1 \Rightarrow A \subseteq G_1 \Rightarrow T \subseteq G_1 \Rightarrow \sigma \in Z_T$ . If, on the other hand,  $\sigma \in Y_{S,A}$ , we also have  $A \subseteq G_1$ , implying  $T \subseteq G_1$ , hence  $\sigma \in Z_T$ .

Next we show that if  $S = T$  and  $B \subseteq A$ , then  $X_{S,A} \subseteq X_{S,B}$ . Clearly, we just need to show that  $Y_{S,A} \subseteq X_{S,B}$ . Take  $\sigma \in Y_{S,A}$ , then we have the following chain of implications:

$$\begin{cases} R_1 = S \\ A \subseteq G_1 \end{cases} \Rightarrow \begin{cases} R_1 = T \\ B \subseteq G_1 \end{cases} \Rightarrow \sigma \in Y_{T,B}.$$

This proves the *if* part of the proposition.

To prove the *only if* part, assume  $X_{S,A} \subseteq X_{T,B}$ . If  $S \neq A$ , set

$$\tau := \begin{array}{|c|c|c|c|c|} \hline \text{supp } \bar{r} & S \setminus A & p_1 & \dots & p_t \\ \hline \emptyset & A & \emptyset & \dots & \emptyset \\ \hline \end{array},$$

else  $S = T$ , and we set

$$\tau := \begin{array}{|c|c|c|c|c|} \hline \text{supp } \bar{r} & p_1 & p_2 & \dots & p_t \\ \hline \emptyset & S & \emptyset & \dots & \emptyset \\ \hline \end{array},$$

where in both cases  $p_1, \dots, p_t$  is a sequence of elements from  $\text{supp } \bar{r} \setminus A$ , with each element  $p$  occurring  $\bar{r}(p)$  times. Clearly, in the first case,  $\tau \in Y_{S,A}$ , and in the second case  $\tau \in Z_S$ , hence  $\tau \in X_{T,B} = Z_T \cup Y_{T,B}$ . This means that either  $T \subseteq A$ , or  $S = T$  and  $B \subseteq A$ .  $\square$

**Lemma 2.8.** Assume  $A \subseteq S \subseteq \text{supp } \bar{r}$  and  $B \subseteq T \subseteq \text{supp } \bar{r}$ . We have

- (1)  $Z_S \cap Z_T = Z_{S \cup T}$ ,
- (2)  $Y_{S,A} \cap Z_T = Y_{S, A \cup T}$ ,
- (3)  $Y_{S,A} \cap Y_{T,B} = \begin{cases} Y_{S, A \cup B}, & \text{if } S = T, \\ \emptyset, & \text{otherwise.} \end{cases}$

**Proof.** To show (1), pick  $\sigma \in Z_S \cap Z_T$ . We have  $S \subseteq G_1$  and  $T \subseteq G_1$ , hence  $S \cup T \subseteq G_1$ , and so  $\sigma \in Z_{S \cup T}$ .

To show (2), pick  $\sigma \in Y_{S,A} \cap Z_T$ . We have  $R_1 = S$ ,  $A \subseteq G_1$ , and  $T \subseteq G_1$ . It follows that  $R_1 = S$  and  $A \cup T \subseteq G_1$ , so  $\sigma \in Y_{S, A \cup T}$ .

Finally, to show (3), pick  $\sigma \in Y_{S,A} \cap Y_{T,B}$ . On one hand,  $\sigma \in Y_{S,A}$  means  $R_1 = S$  and  $A \subseteq G_1$ , on the other hand,  $\sigma \in Y_{T,B}$  means  $R_1 = T$  and  $B \subseteq G_1$ . We conclude that  $Y_{S,A} \cap Y_{T,B} = \emptyset$  if  $S \neq T$ . Otherwise, we have  $R_1 = S = T$  and  $A \cup B \subseteq G_1$ , so  $\sigma \in Y_{S, A \cup B}$ .  $\square$

**Proposition 2.9.** For all pairs of subsets  $A \subseteq S \subseteq \text{supp } \bar{r}$  and  $B \subseteq T \subseteq \text{supp } \bar{r}$  we have the following formulae for the intersection:

$$\begin{aligned} (2.3) \quad & X_{S,A \cup B}, & \text{if } S = T; \\ (2.4) \quad & X_{T, S \cup B}, & \text{if } S \subset T; \\ (2.5) \quad & Z_{S \cup T} = X_{S \cup T, S \cup T}, & \text{if } S \not\subseteq T \text{ and } T \not\subseteq S. \end{aligned}$$

**Proof.** In general, we have

$$\begin{aligned} (2.6) \quad X_{S,A} \cap X_{T,B} &= (Z_S \cap Z_T) \cup (Z_S \cap Y_{T,B}) \cup (Y_{S,A} \cap Z_T) \cup (Y_{S,A} \cap Y_{T,B}) \\ &= \begin{cases} Z_{S \cup T} \cup Y_{T, S \cup B} \cup Y_{S, T \cup A} \cup Y_{S, A \cup B}, & \text{if } S = T; \\ Z_{S \cup T} \cup Y_{T, S \cup B} \cup Y_{S, T \cup A}, & \text{otherwise.} \end{cases} \end{aligned}$$

Assume first that  $S = T$ . In this case  $Y_{T, S \cup B} = Y_{S, T \cup A} = Z_S$ , hence the equation (2.6) translates to  $X_{S,A} \cap X_{T,B} = Z_S \cup Y_{S, A \cup B} = X_{S, A \cup B}$ .

Let us now consider the case  $S \subset T$ . We have  $Y_{S, T \cup A} = \emptyset$ , hence (2.6) translates to  $X_{S,A} \cap X_{T,B} = Z_T \cup Y_{T, S \cup B} = X_{T, S \cup B}$ .

Finally, assume  $S \not\subseteq T$  and  $T \not\subseteq S$ . Then  $Y_{T, S \cup B} = Y_{S, T \cup A} = \emptyset$ , hence (2.6) says  $X_{S,A} \cap X_{T,B} = Z_{S \cup T}$ .  $\square$

For convenience we record the following special cases of Proposition 2.9.

**Corollary 2.10.** For  $S \neq T$  we have

$$X_S \cap X_T = \begin{cases} X_{T,S}, & \text{if } S \subset T, \\ Z_{S \cup T}, & \text{otherwise,} \end{cases}$$

$$(2.7) \quad X_S \cap Z_T = Z_{S \cup T}.$$

**Proof.** The first formula is a simple substitution of  $A = B = \emptyset$  in (2.4) and (2.5). To see (2.7), substitute  $A = \emptyset$ ,  $B = T$  in (2.4) to obtain

$$X_{S, \emptyset} \cap X_{T, T} = \begin{cases} X_{T, S \cup T}, & \text{if } S \subset T \\ Z_{S \cup T}, & \text{otherwise} \end{cases} = \begin{cases} Z_T, & \text{if } S \subset T \\ Z_{S \cup T}, & \text{otherwise} \end{cases} = Z_{S \cup T}. \quad \square$$

We invite the reader to trace the intersections formulae from Corollary 2.10 for the example  $\bar{r} = (2, 1, 1)$ , shown on Figure 2.1.

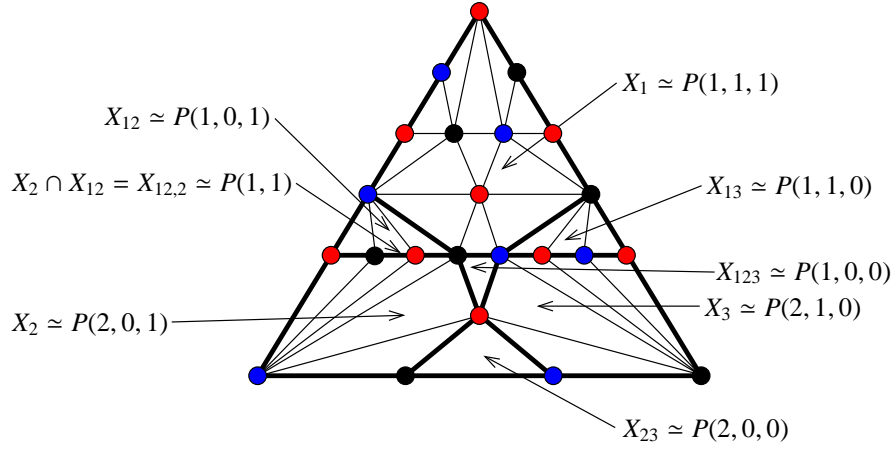


FIGURE 2.1. The canonical decomposition of the immediate snapshot complex  $P(2, 1, 1)$ .

**Remark 2.11.** Corollary 2.10 implies that every stratum  $X_{S,A}$  can be represented as an intersection of two strata of the type  $X_S$ , with only exception provided by the strata  $X_{S,S}$ , when  $|S| = 1$ .

**Corollary 2.12.** Assume  $S_1, \dots, S_t \subseteq [n]$ , such that  $S_1 \not\subseteq S_i$ , for all  $i = 2, \dots, t$ . The following two cases describe the intersection  $X_{S_1} \cap \dots \cap X_{S_t}$ :

- (1) if  $S_1 \supset S_i$ , for all  $i = 2, \dots, t$ , then  $X_{S_1} \cap \dots \cap X_{S_t} = X_{S_1, S_2 \cup \dots \cup S_t}$ ;
- (2) if there exists  $2 \leq i \leq t$ , such that  $S_1 \not\supset S_i$ , then  $X_{S_1} \cap \dots \cap X_{S_t} = Z_{S_1 \cup S_2 \cup \dots \cup S_t} = X_{S_1 \cup S_2 \cup \dots \cup S_t, S_1 \cup S_2 \cup \dots \cup S_t}$ .

**Proof.** Assume first that  $S_1 \supset S_i$ , for all  $i = 2, \dots, t$ . By iterating (2.4) we get

$$\begin{aligned} X_{S_1} \cap \dots \cap X_{S_t} &= X_{S_1, \emptyset} \cap X_{S_2, \emptyset} \cap \dots \cap X_{S_t, \emptyset} = X_{S_1, S_2} \cap X_{S_3, \emptyset} \cap \dots \cap X_{S_t, \emptyset} \\ &= X_{S_1, S_2 \cup S_3} \cap X_{S_4, \emptyset} \cap \dots \cap X_{S_t, \emptyset} = \dots = X_{S_1, S_2 \cup \dots \cup S_t}. \end{aligned}$$

This proves (1).

To show (2), we can assume without loss of generality, that  $S_2 \not\subseteq S_1$ . By (2.5) we have  $X_{S_1} \cap X_{S_2} = Z_{S_1 \cup S_2}$ . By iterating (2.7) we get

$$Z_{S_1 \cup S_2} \cap X_{S_3} \cap \dots \cap X_{S_t} = Z_{S_1 \cup S_2 \cup S_3} \cap X_{S_4} \cap \dots \cap X_{S_t} = X_{S_1 \cup S_2 \cup \dots \cup S_t},$$

which finishes the proof.  $\square$

#### 2.4. The boundary of the immediate snapshot complexes and its canonical decomposition.

**Definition 2.13.** Let  $\bar{r}$  be an arbitrary round counter, and assume  $V \subset \text{supp } \bar{r}$ . We define  $B_V(\bar{r})$  to be the simplicial subcomplex of  $P(\bar{r})$  consisting of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , satisfying  $V \subseteq G_0$ .

The fact that  $B_V(\bar{r})$  is a well-defined subcomplex of  $P(\bar{r})$  is immediate from the definition of the ghosting operation. We shall let  $\beta_V(\bar{r})$  denote the inclusion map

$$\beta_V(\bar{r}) : B_V(\bar{r}) \hookrightarrow P(\bar{r}).$$

**Proposition 2.14.** For an arbitrary round counter  $\bar{r}$ , and any  $V \subset \text{supp } \bar{r}$ , the map  $\delta_V(\bar{r})$  given by

$$\delta_V(\bar{r}) : ((W_0, G_0), \dots, (W_t, G_t)) \mapsto ((W_0, G_0 \setminus V), \dots, (W_t, G_t))$$

is a simplicial isomorphism between simplicial complexes  $B_V(\bar{r})$  and  $P(\bar{r} \setminus V)$ .

**Proof.** The map  $\delta_V(\bar{r})$  is simplicial, and it has a simplicial inverse which adds  $V$  to  $G_0$ .  $\square$

Given an arbitrary round counter  $\bar{r}$ ,  $A \subseteq S \subseteq \text{act } \bar{r}$ , and  $V \subset \text{supp } \bar{r}$ , such that  $S \cap V = \emptyset$ , we set

$$X_{S,A,V}(\bar{r}) := X_{S,A}(\bar{r}) \cap B_V(\bar{r}).$$

We can use the notational convention  $B_\emptyset(\bar{r}) = P(\bar{r})$ , which is consistent with Definition 2.13. In this case we get  $X_{S,A,\emptyset}(\bar{r}) = X_{S,A}(\bar{r})$ , fitting well with the previous notations.

The diagram (2.8) in the next proposition means that we can naturally think about  $X_{S,A,V}(\bar{r})$  both as  $X_{S,A}(\bar{r} \setminus V)$  as well as  $B_V(\bar{r}_{S,A})$ , or abusing notations we write  $B_V \cap X_{S,A} = X_{S,A}(B_V) = B_V(X_{S,A})$ .

**Proposition 2.15.** Assume  $\bar{r}$  is an arbitrary round counter,  $V \subset \text{supp } \bar{r}$ ,  $A \subseteq S \subseteq \text{act } \bar{r}$ , and  $V \cap S = \emptyset$ . Then there exist simplicial isomorphisms  $\varphi$  and  $\psi$  making the following diagram commute:

$$(2.8) \quad \begin{array}{ccccc} P(\bar{r}) & \xleftarrow{\alpha} & X_{S,A}(\bar{r}) & \overset{\gamma}{\rightsquigarrow} & P(\bar{r}_{S,A}) \\ \uparrow \beta & & \uparrow j & & \uparrow \beta \\ B_V(\bar{r}) & \xleftarrow{i} & X_{S,A,V}(\bar{r}) & \overset{\varphi}{\rightsquigarrow} & B_V(\bar{r}_{S,A}) \\ \downarrow \delta & & \downarrow \psi & & \downarrow \delta \\ P(\bar{r} \setminus V) & \xleftarrow{\alpha} & X_{S,A}(\bar{r} \setminus V) & \overset{\gamma}{\rightsquigarrow} & P(\bar{r}_{S \cup V, A \cup V}), \end{array}$$

where  $i$  and  $j$  denote inclusion maps.

**Proof.** Note that  $X_{S,A,V}(\bar{r})$  consists of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $V \subseteq G_0$ ,  $A \subseteq G_1$ , and either  $W_1 \cup G_1 = S$ , or  $S \subseteq G_1$ . The fact that  $V$  and  $S$  are disjoint ensures that these conditions do not contradict each other. We let  $\varphi$  be the restriction of  $\gamma_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \rightarrow P(\bar{r}_{S,A})$  to  $X_{S,A,V}(\bar{r})$ . Furthermore, we let  $\psi$  be the restriction of  $\delta_V(\bar{r}) : B_V(\bar{r}) \rightarrow P(\bar{r} \setminus V)$  to  $X_{S,A,V}(\bar{r})$ .  $\square$

The commuting diagram (2.9) in the next proposition shows how the stratum  $X_{S,A}(\bar{r})$  can be naturally interpreted as a part of the boundary of the stratum  $X_{S,B}(\bar{r})$ , whenever  $B \subseteq A \subseteq S \subseteq \text{act } \bar{r}$ .

**Proposition 2.16.** *Assume  $B \subseteq A \subseteq S \subseteq \text{act } \bar{r}$ , then the following diagram commutes*

$$(2.9) \quad \begin{array}{ccccc} X_{S,B}(\bar{r}) & \xleftarrow{i} & X_{S,A}(\bar{r}) & & \\ \gamma_{S,B}(\bar{r}) \downarrow & & \gamma_{S,A}(\bar{r}) \downarrow & & \\ P(\bar{r}_{S,B}) & \xleftarrow{\beta_{A \setminus B}(\bar{r}_{S,B})} & B_{A \setminus B}(\bar{r}_{S,B}) & \xrightarrow{\delta_{A \setminus B}(\bar{r}_{S,B})} & P(\bar{r}_{S,A}) \end{array}$$

where  $i$  denotes the inclusion map.

**Proof.** Take  $\sigma = ((W_0, G_0), \dots, (W_t, G_t)) \in X_{S,A}(\bar{r})$ . On one hand we have

$$(\gamma_{S,B}(\bar{r}) \circ i)(\sigma) = \begin{cases} \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 \setminus B & G_2 & \dots & G_t \\ \hline \end{array} & \text{if } W_1 \cup G_1 = S, A \subseteq G_1; \\ \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S \setminus B & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array} & \text{if } S \subseteq G_1. \end{cases}$$

On the other hand, we have

$$(\gamma_{S,A}(\bar{r}))(\sigma) = \begin{cases} \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array} & \text{if } W_1 \cup G_1 = S, A \subseteq G_1; \\ \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S \setminus A & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array} & \text{if } S \subseteq G_1. \end{cases}$$

Since applying  $\delta_{A \setminus B}(\bar{r}_{S,B})^{-1}$  will add  $A \setminus B$  to  $G_0 \cup G_1 \setminus A$ , resp.  $G_0 \cup S \setminus A$ , above and  $A \subseteq S, A \subseteq G_1$ , we conclude that

$$(\gamma_{S,B}(\bar{r}) \circ i)(\sigma) = (\beta_{A \setminus B}(\bar{r}_{S,B}) \circ \delta_{A \setminus B}(\bar{r}_{S,B})^{-1} \circ \gamma_{S,A}(\bar{r}))(\sigma).$$

Which is the same as to say that the diagram (2.9) commutes.  $\square$

### 2.5. The combinatorial structure of the complexes $P(\chi_{A,B})$ .

Let us analyze the simplicial structure of  $P(\chi_{A,B})$ . Set  $k := |A| - 1$  and  $m := |B|$ . By (1.6) the simplicial complex  $P(\chi_{A,B})$  is isomorphic to the  $m$ -fold suspension of  $P(\chi_A)$ . On the other hand, we saw before that  $P(\chi_A)$  is isomorphic to the standard chromatic subdivision of  $\Delta^k$ . The simplices of the  $m$ -fold suspension of  $\chi(\Delta^k)$  (which is of course homeomorphic to  $\Delta^{m+k}$ ) are indexed by tuples  $(S, (B_1, \dots, B_t)(C_1, \dots, C_t))$ , where  $S$  is any subset of  $B$ , and the sets  $B_1, \dots, B_t, C_1, \dots, C_t$  satisfy the same conditions as in the combinatorial description of the simplicial structure of  $\chi(\Delta^k)$ . In line with (1.7), the simplicial isomorphism between  $P(\chi_{A,B})$  and the  $m$ -fold suspension of  $\chi(\Delta^k)$  can be explicitly given by

$$(S, (B_1, \dots, B_t)(C_1, \dots, C_t)) \mapsto \begin{array}{|c|c|c|c|} \hline W_0 & C_1 & \dots & C_t \\ \hline (A \cup B) \setminus W_0 & B_1 \setminus C_1 & \dots & B_t \setminus C_t \\ \hline \end{array},$$

where  $W_0 = S \cup B_1 \cup \dots \cup B_t$ . In particular, up to the simplicial isomorphism, the complex  $P(\chi_{A,B})$  depends only on  $m$  and  $k$ .

The simplices of  $P(\chi_{A,B})$  are indexed by all witness structures  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  satisfying the following conditions:

- (1)  $W_0 \cup G_0 = A \cup B$ ;
- (2)  $W_0 \cap A = W_1 \cup \dots \cup W_t \cup G_1 \cup \dots \cup G_t$ ;
- (3) the sets  $W_1, \dots, W_t, G_1, \dots, G_t$  are disjoint.

It was shown in [Ko12] that there is a homeomorphism

$$\tau_A : P(\chi_A) \xrightarrow[\cong]{} \Delta^A,$$

such that for any  $C \subseteq A$  the following diagram commutes

$$(2.10) \quad \begin{array}{ccccc} P(\chi_A) & \xleftarrow{\beta_{A \setminus C}(\chi_A)} & B_{A \setminus C}(\chi_A) & \xrightarrow{\delta_{A \setminus C}(\chi_A)} & P(\chi_C) \\ \tau_A \downarrow \cong & & & & \nearrow \tau_C \cong \\ \Delta^A & \xleftarrow{i} & \Delta^C & & \end{array}$$

where  $i : \Delta^C \hookrightarrow \Delta^A$  is the standard inclusion map. In general, given a pair of sets  $(A, B)$ , we take the  $|B|$ -fold suspension of the map  $\tau_A$  to produce a homeomorphism

$$\tau_{A,B} : P(\chi_{A,B}) \xrightarrow[\cong]{} \Delta^{A \cup B}.$$

**Definition 2.17.** When  $A \cup B = C \cup D$ , we set

$$\tau(\chi_{A,B}, \chi_{C,D}) := \tau_{C,D}^{-1} \circ \tau_{A,B},$$

clearly, we get a homeomorphism  $\tau(\chi_{A,B}, \chi_{C,D}) : P(\chi_{A,B}) \xrightarrow[\cong]{} P(\chi_{C,D})$ .

We know that this map is a simplicial isomorphism when restricted to  $B_S(\chi_{A,B})$ , for all  $S \subseteq (A \cap C) \cup (B \cap D)$ , i.e., we have the following commutative diagram

$$(2.11) \quad \begin{array}{ccc} B_S(\chi_{A,B}) & \xrightarrow{\tau(\chi_{A,B}, \chi_{C,D})} & B_S(\chi_{C,D}) \\ \downarrow \beta_S(\chi_{A,B}) & & \downarrow \beta_S(\chi_{C,D}) \\ P(\chi_{A,B}) & \xrightarrow[\cong]{\tau(\chi_{A,B}, \chi_{C,D})} & P(\chi_{C,D}) \end{array}$$

When  $C \subseteq A$ , we have  $B \subseteq D$ , so the condition for  $S$  becomes  $S \subseteq B \cup C$ . Furthermore, if in addition  $T = E \cup F$ , we have

$$\tau(\chi_{A_1, B_1}, \chi_{A_2, B_2}) \circ \tau(\chi_{A_2, B_2}, \chi_{A_3, B_3}) = \tau(\chi_{A_1, B_1}, \chi_{A_3, B_3}).$$

When  $A \subseteq C \cup D$ , the identity (1.1) implies that we have a simplicial isomorphism

$$\beta_V(\chi_{C,D}) : B_V(\chi_{C,D}) \xrightarrow[\cong]{} P(\chi_{C \setminus A, D \setminus A}).$$

Furthermore, when  $S \subseteq C$ , the identity (1.4) implies that we have a simplicial isomorphism

$$X_S(\chi_{C,D}) : \gamma_S(\chi_{C,D}) \xrightarrow[\cong]{} P(\chi_{C \setminus S, D \cup S}).$$

**Proposition 2.18.** Assume  $A \cup B = C \cup D$  and  $V \subseteq A \cup B$ , then the following diagram commutes

$$(2.12) \quad \begin{array}{ccccc} P(\chi_{A,B}) & \xleftarrow{\beta_V(\chi_{A,B})} & B_V(\chi_{A,B}) & \xrightarrow{\delta_V(\chi_{A,B})} & P(\chi_{A,B} \setminus V) \\ \tau(\chi_{A,B}, \chi_{C,D}) \downarrow \cong & & & & \downarrow \cong \tau(\chi_{A,B} \setminus V, \chi_{C,D} \setminus V) \\ P(\chi_{C,D}) & \xleftarrow{\beta_V(\chi_{C,D})} & B_V(\chi_{C,D}) & \xrightarrow{\delta_V(\chi_{C,D})} & P(\chi_{C,D} \setminus V) \end{array}$$

**Proof.** Consider the diagram on Figure 2.2. Both the upper and the lower part of this diagram are versions of (2.10), hence, they commute. Together, they form the diagram (2.12).  $\square$

$$\begin{array}{ccccc}
P(\chi_{A,B}) & \xleftarrow{\beta_V(\chi_{A,B})} & B_V(\chi_{A,B}) & \xrightarrow{\delta_V(\chi_{A,B})} & P(\chi_{A,B} \setminus V) \\
\tau_{A,B} \downarrow \cong & & & & \nwarrow \tau_{A \setminus V, B \setminus V} \cong \\
\Delta^{A \cup B} & \xleftarrow{\quad} & \Delta^{A \cup B \setminus V} & & \\
\tau_{C,D} \uparrow \cong & & & & \nearrow \tau_{C \setminus V, D \setminus V} \cong \\
P(\chi_{C,D}) & \xleftarrow{\beta_V(\chi_{C,D})} & B_V(\chi_{C,D}) & \xrightarrow{\delta_V(\chi_{C,D})} & P(\chi_{C,D} \setminus V)
\end{array}$$

FIGURE 2.2. Commuting diagram used in the proof of Proposition 2.18

### 3. TOPOLOGY OF THE IMMEDIATE SNAPSHOT COMPLEXES

#### 3.1. Immediate snapshot complexes are collapsible pseudomanifolds.

Consider a quite general situation, where  $X$  is an arbitrary topological space, and  $\{X_i\}_{i \in I}$  is a finite family of subspace of  $X$  covering  $X$ , that is  $I$  is finite and  $X = \bigcup_{i \in I} X_i$ .

**Definition 3.1.** ([Ko07, Definition 15.14]). *The nerve complex  $\mathcal{N}$  of a covering  $\{X_i\}_{i \in I}$  is a simplicial complex whose vertices are indexed by  $I$ , and a subset of vertices  $J \subseteq I$  spans a simplex if and only if the intersection  $\bigcap_{i \in J} X_i$  is not empty.*

The nerve complex can be useful because of the following fact.

**Lemma 3.2.** (Nerve Lemma, [Ko07, Theorem 15.21, Remark 15.22]). *Assume  $K$  is a simplicial complex, covered by a family of subcomplexes  $\mathcal{K} = \{K_i\}_{i \in I}$ , such that  $\bigcap_{i \in J} K_i$  is empty or contractible for all  $J \subseteq I$ , then  $K$  is homotopy equivalent to the nerve complex  $\mathcal{N}(\mathcal{K})$ .*

**Corollary 3.3.** *For an arbitrary round counter  $\bar{r}$ , the simplicial complex  $P(\bar{r})$  is contractible.*

**Proof.** We use induction on  $|\bar{r}|$ . If  $|\bar{r}| = 0$ , then  $P(\bar{r})$  is just a simplex, hence contractible. We assume that  $|\bar{r}| \geq 1$ , and view the canonical decomposition  $P(\bar{r}) = \bigcup_{S \subseteq \text{act } \bar{r}} X_S(\bar{r})$  as a covering of  $P(\bar{r})$ . By Proposition 2.4, Corollary 2.12, and the induction assumption, all the intersections of the subcomplexes  $X_S(\bar{r})$  with each other are either empty or contractible. This means, that we can apply the Nerve Lemma 3.2, with  $K = P(\bar{r})$ ,  $I = 2^{\text{act } \bar{r}} \setminus \{\emptyset\}$ , and  $K_i$ 's are  $X_S(\bar{r})$ 's.

Now, by Corollary 2.12 we see that  $X_{\text{act } \bar{r}} \cap X_S = X_{\text{act } \bar{r}, S} \neq \emptyset$  for all  $S \subset \text{act } \bar{r}$ . It follows that the nerve complex of this decomposition as a cone with apex at  $\text{act } \bar{r} \in I$ . Since the nerve complex is contractible, it follows from the Nerve Lemma 3.2 that  $P(\bar{r})$  is contractible as well.  $\square$

While contractibility is a property of topological spaces, there is a stronger combinatorial property called *collapsibility*, see [Co73], which some simplicial complexes may have.

**Definition 3.4.** *Let  $K$  be a simplicial complex. A pair of simplices  $(\sigma, \tau)$  of  $K$  is called an elementary collapse if the following conditions are satisfied:*

- $\tau$  is a maximal simplex,
- $\tau$  is the only simplex which properly contains  $\sigma$ .

A finite simplicial complex  $K$  is called **collapsible**, if there exists a sequence  $(\sigma_1, \tau_1), \dots, (\sigma_t, \tau_t)$  of pairs of simplices of  $K$ , such that

- this sequence yields a perfect matching on the set of all simplices of  $K$ ,



- for every  $1 \leq k \leq t$ , the pair  $(\sigma_k, \tau_k)$  is an elementary collapse in  $K \setminus \{\sigma_1, \dots, \sigma_{k-1}, \tau_1, \dots, \tau_{k-1}\}$ .

When  $(\sigma, \tau)$  is an elementary collapse, we also say that  $\sigma$  is a *free simplex*.

We have shown in Proposition 1.23 that for any round counter  $\bar{r}$  the simplicial complex  $P(\bar{r})$  is a pseudomanifold with boundary  $\partial P(\bar{r})$ . Set

$$\text{int } P(\bar{r}) := \bigcup_{\sigma \in P(\bar{r}), \sigma \notin \partial P(\bar{r})} \text{int } \sigma,$$

and, for all  $A \subseteq S \subseteq \text{act } \bar{r}$ , set

$$\partial X_{S,A}(\bar{r}) := \gamma_{S,A}(\bar{r})^{-1}(\partial P(\bar{r}_{S,A})), \quad \text{int } X_{S,A}(\bar{r}) := \gamma_{S,A}(\bar{r})^{-1}(\text{int } P(\bar{r}_{S,A})).$$

**Proposition 3.5.** *Assume  $\bar{r}$  is an arbitrary round counter,  $A \subset S \subseteq \text{act } \bar{r}$ , and  $V \subseteq \text{supp } \bar{r} \setminus S$ . The simplicial complex  $\partial X_{S,A,V}(\bar{r})$  is the subcomplex of  $X_{S,A,V}(\bar{r})$  consisting of all simplices*

$$\sigma = ((W_0, V), (S \setminus A, A), \dots, (W_t, G_t)).$$

**Proof.** Pick  $\sigma \in X_{S,A,V}$ , and set  $\rho$  to be the composition of the simplicial isomorphisms  $X_{S,A,V}(\bar{r}) \rightarrow B_V(\bar{r}_{S,A}) \rightarrow P(\bar{r}_{S \cup V, A \cup V})$  from the commutative diagram (2.8).

Assume first that  $W_1 \cup G_1 = S$ , then

$$\rho(\sigma) = ((W_0 \setminus G_1, (G_0 \cup G_1) \setminus (A \cup V)), (W_2, G_2), \dots, (W_t, G_t)).$$

Clearly  $\rho(\sigma) \notin \partial P(\bar{r}_{S \cup V, A \cup V})$  if and only if  $(G_0 \cup G_1) \setminus (A \cup V) = \emptyset$ , i.e.,  $G_0 \cup G_1 \subseteq A \cup V$ . Since we know that  $A \subseteq G_1$ ,  $V \subseteq G_0$ , this means that  $G_0 = V$  and  $G_1 = A$ , which implies  $W_1 = S \setminus A$ .

Assume now that  $S \subseteq G_1$ , then we have

$$\rho(\sigma) = ((W_0 \setminus S, (G_0 \cup S) \setminus (A \cup V)), (W_1, G_1 \setminus S), (W_2, G_2), \dots, (W_t, G_t)).$$

Here we have  $\rho(\sigma) \notin \partial P(\bar{r}_{S \cup V, A \cup V})$  if and only if  $(G_0 \cup S) \setminus (A \cup V) = \emptyset$ , which is impossible, since  $V \cap S = \emptyset$ , and  $A \subset V$ .  $\square$

**Corollary 3.6.** *The simplicial complex  $P(\bar{r})$  can be decomposed as a disjoint union of the simplex  $\Delta^{\text{pass } \bar{r}} = ((\text{pass } \bar{r}, \text{act } \bar{r}))$ , and the sets  $\text{int } X_{S,A,V}$ , where  $(S, A, V)$  range over all triples satisfying  $A \subset S \subseteq \text{act } \bar{r}$  and  $V \subseteq \text{supp } \bar{r} \setminus S$ .*

*Specifically, for a simplex  $\sigma \in P(\bar{r})$ ,  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , we have: if  $t = 0$ , then  $\sigma \subseteq \Delta^{\text{pass } \bar{r}}$ , else  $\text{int } \sigma \subseteq \text{int } X_{W_1 \setminus G_1, G_1, G_0}$ .*

**Proof.** Immediate from Proposition 3.5.  $\square$

**Lemma 3.7.** *Assume  $\bar{r}$  is a round counter, and  $p \in \text{supp } \bar{r}$ , then there exists a sequence of elementary collapses reducing the simplicial complex  $P(\bar{r})$  to the subcomplex  $(\partial P(\bar{r})) \setminus \text{int } B_p(\bar{r})$ .*

**Proof.** The proof is again by induction on  $|r|$ . The case  $|r| = 0$  is trivial. The simplices we need to collapse are precisely those, whose interior lies in  $\text{int } P(\bar{r}) \cup \text{int } B_p(\bar{r})$ . Let  $\Sigma$  denote the set of all strata  $X_{S,A}$ , where  $A \subset S \subseteq \text{act } \bar{r}$ , together with all strata  $X_{S,A,p}$ , where  $A \subset S \subseteq \text{act } \bar{r}$ ,  $p \notin S$ . By Corollary 3.6, the union of the interiors of the strata in  $\Sigma$  is precisely  $\text{int } P(\bar{r}) \cup \text{int } B_p(\bar{r})$ .

We describe our collapsing as a sequence of steps. At each step we pick a certain pair of strata  $(Y, X)$ , where  $Y \subset X$ , which we must “collapse”. Then, we use one of the previous results to show that as a simplicial pair  $(Y, X)$  is isomorphic to  $(B_t(\bar{r}'), P(\bar{r}'))$ , for some round counter  $\bar{r}'$ , such that  $|\bar{r}'| < |\bar{r}|$ . By induction assumption this means that there is a sequence of simplicial collapses which removes  $\text{int } X \cup \text{int } Y$ . Finally, we order these

pairs of strata with disjoint interiors  $(Y_1, X_1), \dots, (Y_d, X_d)$  such that for every  $1 \leq i \leq d$ , every simplex  $\sigma \in P(\bar{r})$ , such that  $\text{int } \sigma \subseteq \text{int } X_i \cup \text{int } Y_i$ , and every  $\tau \supset \sigma$ , such that  $\dim \tau = \dim \sigma + 1$ , we have

$$(3.1) \quad \text{int } \tau \subseteq \text{int } X_1 \cup \dots \cup \text{int } X_i \cup \text{int } Y_1 \cup \dots \cup \text{int } Y_i.$$

This means, that at step  $i$  we can collapse away the pair of strata  $(Y_i, X_i)$  (i.e., collapse away those simplices whose interior is contained in  $\text{int } X_i \cup \text{int } Y_i$ ) using the procedure given by the induction assumption, and that these elementary collapses will be legal in  $P(\bar{r}) \setminus (X_1 \cup \dots \cup \text{int } X_{i-1} \cup \text{int } Y_1 \cup \dots \cup \text{int } Y_{i-1})$  as well.

Our procedure is now divided into 3 stages. At stage 1, we match the strata  $X_{S,A,p}$  with  $X_{S,A}$ , for all  $A \subset S \subseteq \text{act } \bar{r}$ , such that  $p \notin S$ . It follows from the commutativity of the diagram (2.8) that each pair of simplicial subcomplexes  $(X_{S,A,p}, X_{S,A})$  is isomorphic to the pair  $(B_p(\bar{r}_{S,A}), P(\bar{r}_{S,A}))$ . We have  $|\bar{r}_{S,A}| \leq |\bar{r}| - |S| < |\bar{r}|$ , hence by induction assumption, this pair can be collapsed. As a collapsing order we choose any order which does not decrease the cardinality of the set  $A$ . Take  $\sigma$  such that  $\text{int } \sigma \subseteq \text{int } X_{S,A,p} \cup \text{int } X_{S,A}$ . By Proposition 3.5 this means that  $\sigma = ((W_0, T), (S \setminus A, A), \dots)$ , where either  $T = \emptyset$ , or  $T = \{p\}$ . Take  $\tau \supset \sigma$ , such that  $\dim \tau = \dim \sigma + 1$ . Then by Proposition 1.17(b) there exists  $q \in A(\tau)$ , such that  $\sigma = \Gamma_q(\tau)$ . A case-by-case analysis of the ghosting construction shows that  $\text{int } \tau \subseteq \text{int } X$ , where  $X$  is one of the following strata:  $X_{S,A}, X_{S,A,p}, X_q, X_{q,\emptyset,p}, X_{S,A \setminus \{q\}}, X_{S,A \setminus \{q\},p}$ . Since the order in which we do collapses does not decrease the cardinality of  $A$ , the interiors of the last 4 of these strata have already been removed, hence the condition (3.1) is satisfied.

At stage 2, we match  $X_S$  with  $X_{S,S \setminus \{p\}}$ , for all  $S \subseteq \text{act } \bar{r}$ , such that  $p \in S$ ,  $|S| \geq 2$ . By commutativity of the diagram (2.9), the pair  $(X_{S,S \setminus \{p\}}, X_S)$  is isomorphic to  $(B_{S \setminus \{p\}}(\bar{r}_S), P(\bar{r}_S))$ . This big collapse can easily be expressed as a sequence of elementary collapses, though in a non-canonical way. For this, we pick any  $q \in S \setminus \{p\}$ . It exists, since we assumed that  $|S| \geq 2$ . Then we match pairs  $(X_{S,A \cup \{q\}}, X_{S,A})$ , for all  $A \subseteq S \setminus \{q\}$ . Again, by commutativity of the diagram (2.9), this pair is isomorphic to  $(B_q(\bar{r}_{S,A}), P(\bar{r}_{S,A}))$ . The order in which we arrange  $S$  does not matter for the collapsing order. Once  $S$  is fixed, the collapsing order inside does not decrease the cardinality of  $A$ . As above, take  $\sigma$  such that  $\text{int } \sigma \subseteq \text{int } X_{S,A \cup \{q\}} \cup \text{int } X_{S,A}$ , take  $\tau \supset \sigma$ , such that  $\dim \tau = \dim \sigma + 1$ , and take  $r \in A(\tau)$ , such that  $\sigma = \Gamma_q(\tau)$ . By Proposition 3.5 we have  $\sigma = ((W_0, \emptyset), (S \setminus A, A), \dots)$ , or  $\sigma = ((W_0, \emptyset), (S \setminus (A \cup \{q\}), A \cup \{q\}), \dots)$ . Note, that both  $q$  and  $r$  are different from  $p$ , but we may have  $q = r$ . Again, a case-by-case analysis of the ghosting construction shows that  $\text{int } \tau \subseteq \text{int } X$ , where  $X$  is one of the following strata:  $X_{S,A}, X_{S,A \cup \{q\}}, X_{S,A \setminus \{r\}}, X_{S,A \cup \{q\} \setminus \{r\}}, X_q, X_r$ . Again, since collapsing order does not decrease the cardinality of  $A$ , the condition (3.1) is satisfied.

At stage 3, we collapse the pair  $(X_{p,p}, X_p)$ . Let us be specific. First, by Corollary 2.6 we know that  $X_{p,p} = \bigcup_{\{p\} \subset S \subseteq \text{act } \bar{r}} X_{S,p}$ , and it follows from Proposition 3.5 that  $\text{int } X_{p,p} = \bigcup_{\{p\} \subset S \subseteq \text{act } \bar{r}} \text{int } X_{S,p}$ . By commutativity of the diagram (2.9), the pair  $(X_{p,p}, X_p)$  is isomorphic to  $(B_p(\bar{r}_p), P(\bar{r}_p))$ , hence it can be collapsed using the induction assumption. Clearly, the entire procedure exhausts the set  $\Sigma$ , and we arrive at the simplicial complex  $(\partial P(\bar{r})) \setminus \text{int } B_p(\bar{r})$ .  $\square$

**Corollary 3.8.** *For an arbitrary round counter  $\bar{r}$ , the simplicial complex  $P(\bar{r})$  is collapsible.*

**Proof.** Iterative use of Lemma 3.7.  $\square$

### 3.2. Homeomorphic gluing.

**Definition 3.9.** We say that a simplicial complex  $K$  is **simplicially homeomorphic** to a simplex  $\Delta^A$ , where  $A$  is some finite set, if there exists a homeomorphism  $\varphi : \Delta^A \rightarrow K$ , such that for every simplex  $\sigma \in \Delta^A$ , the image  $\varphi(\sigma)$  is a subcomplex of  $K$ .

When we say that a CW complex is *finite* we shall mean that it has finitely many cells.

**Definition 3.10.** Let  $X$  and  $Y$  be finite CW complexes. A **homeomorphic gluing data** between  $X$  and  $Y$  consists of the following:

- a family  $(A_i)_{i=1}^t$  of CW subcomplexes of  $X$ , such that  $X = \cup_{i=1}^t A_i$ ,
- a family  $(B_i)_{i=1}^t$  of CW subcomplexes of  $Y$ , such that  $Y = \cup_{i=1}^t B_i$ ,
- a family of homeomorphisms  $(\varphi_i)_{i=1}^t$ ,  $\varphi_i : A_i \rightarrow B_i$ ,

satisfying the compatibility condition: if  $x \in A_i \cap A_j$ , then  $\varphi_i(x) = \varphi_j(x)$ .

Given finite CW complexes  $X$  and  $Y$ , together with homeomorphic gluing data  $(A_i, B_i, \varphi_i)_{i=1}^t$  from  $X$  to  $Y$ , we define  $\varphi : X \rightarrow Y$ , by setting  $\varphi(x) := \varphi_i(x)$ , whenever  $x \in A_i$ . The compatibility condition from Definition 3.10 implies that  $\varphi(x)$  is independent of the choice of  $i$ , hence the map  $\varphi : X \rightarrow Y$  is well-defined.

**Lemma 3.11.** (Homeomorphism Gluing Lemma).

Assume we are given finite CW complexes  $X$  and  $Y$ , and homeomorphic gluing data  $(A_i, B_i, \varphi_i)_{i=1}^t$ , satisfying an additional condition:

$$(3.2) \quad \text{if } \varphi(x) \in B_i, \text{ then } x \in A_i,$$

then the map  $\varphi : X \rightarrow Y$  is a homeomorphism.

**Proof.** First it is easy to see that  $\varphi$  is surjective. Take an arbitrary  $y \in Y$ , then there exists  $i$  such that  $y \in B_i$ . Take  $x = \varphi_i^{-1}(y)$ , clearly  $\varphi(x) = y$ .

Let us now check the injectivity of  $\varphi$ . Take  $x_1, x_2 \in X$  such that  $\varphi(x_1) = \varphi(x_2)$ . There exists  $i$  such that  $x_1 \in A_i$ . Then  $\varphi(x_1) = \varphi_i(x_1) \in B_i$ , hence  $\varphi(x_2) \in B_i$ . Condition (3.2) implies that  $x_2 \in A_i$ . The fact that  $x_1 = x_2$  now follows from the injectivity of  $\varphi_i$ .

We have verified that  $\varphi$  is bijective, so  $\varphi^{-1} : Y \rightarrow X$  is a well-defined map. We shall now prove that  $\varphi^{-1}$  is continuous by showing that  $\varphi$  takes closed sets to closed sets. To start with, let us recall the following basic property of the topology of CW complexes: a subset  $A$  of a CW complex  $X$  is closed if and only if its intersection with the closure of each cell in  $X$  is closed. Sometimes, one uses the terminology *weak topology* of the CW complex. This property was an integral part of the original J.H.C. Whitehead definition of CW complexes, see, e.g., [Hat02, Proposition A.2.] for further details.

Let us return to our situation. We claim that  $A \subseteq X$  is closed, if and only if  $A \cap A_i$  is closed in  $A_i$ , for each  $i = 1, \dots, t$ . Note first that since  $A_i$  is itself closed, a subset  $S \subseteq A_i$  is closed in  $X$  if and only if it is closed in  $A_i$ , so we will skip mentioning where the sets are closed. Clearly, if  $A$  is closed, then  $A \cap A_i$  is closed for all  $i = 1, \dots, t$ . On the other hand, assume  $A \cap A_i$  is closed for all  $i$ . Let  $\sigma$  be a closed cell of  $X$ , we need to show that  $A \cap \sigma$  is closed. Since  $X = \cup_{i=1}^t A_i$ , and  $A_i$ 's are CW subcomplexes of  $X$ , there exists  $i$ , such that  $\sigma \subseteq A_i$ . Then  $A \cap \sigma = A \cap (A_i \cap \sigma) = (A \cap A_i) \cap \sigma$ , but  $(A \cap A_i) \cap \sigma$  is closed since  $A \cap A_i$  is closed. Hence  $A \cap \sigma$  is closed and our argument is finished. Similarly, we can show that  $B \subseteq X$  is closed, if and only if  $B \cap B_i$  is closed, for each  $i = 1, \dots, t$ .

Pick now a closed set  $A \subseteq X$ , we want to show that  $\varphi(A)$  is closed. To start with, for all  $i$  the set  $A \cap A_i$  is closed, hence  $\varphi_i(A \cap A_i) \subseteq B_i$  is also closed, since  $\varphi_i$  is a homeomorphism. Let us verify that for all  $i$  we have

$$(3.3) \quad \varphi_i(A \cap A_i) = \varphi(A) \cap B_i.$$

Assume  $y \in \varphi_i(A \cap A_i)$ . On one hand  $y \in \varphi_i(A_i)$ , so  $y \in B_i$ , on the other hand,  $y = \varphi_i(x)$ , for  $x \in A$ , so  $y \in \varphi(A)$ . Reversely, assume  $y \in \varphi(A)$  and  $y \in B_i$ . Then  $y = \varphi(x) \in B_i$ , so condition (3.2) implies that  $x \in A_i$ , hence  $y \in \varphi(A \cap A_i)$ , which proves (3.3). It follows that  $\varphi(A) \cap B_i$  is closed for all  $i$ , hence  $\varphi(A)$  itself is closed. This proves that  $\varphi^{-1}$  is continuous.

We have now shown that  $\varphi^{-1} : Y \rightarrow X$  is a continuous bijection. Since  $X$  and  $Y$  are both finite CW complexes, they are compact Hausdorff when viewed as topological spaces. It is a basic fact of set-theoretic topology that a continuous bijection between compact Hausdorff topological spaces is automatically a homeomorphism, see e.g., [Mun, Theorem 26.6].  $\square$

The following variations of the Homeomorphism Gluing Lemma 3.11 will be useful for us.

**Corollary 3.12.** *Assume we are given finite CW complexes  $X$  and  $Y$ , and homeomorphic gluing data  $(A_i, B_i, \varphi_i)_{i=1}^t$ , satisfying an additional condition:*

$$(3.4) \quad \text{for all } I \subseteq [t] : \varphi : A_I \rightarrow B_I \text{ is a bijection.}$$

*Then the map  $\varphi : X \rightarrow Y$  is a homeomorphism.*

**Proof.** Clearly, we just need to show that the condition (3.4) implies the condition (3.2). Assume  $y = \varphi(x)$ ,  $y \in B_i$ , and  $x \notin A_i$ . Let  $I$  be the maximal set such that  $y \in B_I$ . The condition (3.4) implies that there exists a unique element  $\tilde{x} \in A_I$ , such that  $\varphi(\tilde{x}) = y$ . In particular,  $\tilde{x} \in A_i$ , hence  $x \neq \tilde{x}$ . Even stronger, if  $x \in A_i$ , for some  $i \in I$ , then  $x, \tilde{x} \in A_i$ , hence  $x = \tilde{x}$ , since  $\varphi_i$  is injective. So  $x_i \notin A_i$ , for all  $i \in I$ . Hence, there exists  $j \notin I$ , such that  $x \in A_j$ , which implies  $\varphi(x) \in B_j$ , yielding a contradiction to the maximality of the set  $I$ .  $\square$

**Corollary 3.13.** *Assume we are given CW complexes  $X$  and  $Y$ , a collection  $(A_i)_{i=1}^t$  of CW subcomplexes of  $X$ , a collection  $(B_i)_{i=1}^t$  of CW subcomplexes of  $Y$ , and a collection  $(\varphi_i)_{i \in [t]}$  of maps such that*

- $X = \cup_{i=1}^t A_i$ ,  $Y = \cup_{i=1}^t B_i$ ;
- for every  $I \subseteq [t]$ , the map  $\varphi_I : A_I \rightarrow B_I$  is a homeomorphism;
- for every  $J \supseteq I$  the following diagram commutes

$$(3.5) \quad \begin{array}{ccc} A_J & \xrightarrow[\cong]{\varphi_J} & B_J \\ \downarrow & & \downarrow \\ A_I & \xrightarrow[\cong]{\varphi_I} & B_I \end{array}$$

*Then  $(A_i, B_i, \varphi_i)_{i=1}^t$  is a homeomorphic gluing data, and the map  $\varphi : X \rightarrow Y$  defined by this data is a homeomorphism.*

**Proof.** For arbitrary  $1 \leq i, j \leq t$ , commutativity of (3.5) implies that also the following diagram is commutative

$$\begin{array}{ccccc} A_i & \longleftrightarrow & A_{\{i,j\}} & \longleftrightarrow & A_j \\ \cong \downarrow \varphi_{\{i\}} & & \cong \downarrow \varphi_{\{i,j\}} & & \cong \downarrow \varphi_{\{j\}} \\ B_i & \longleftrightarrow & B_{\{i,j\}} & \longleftrightarrow & B_j \end{array}$$

In other words, for any  $x \in A_i \cap A_j$ , we have  $\varphi_{\{i\}}(x) = \varphi_{\{i,j\}}(x) = \varphi_{\{j\}}(x)$ . It follows that  $(A_i, B_i, \varphi_{\{i\}})_{i=1}^t$  is a homeomorphic gluing data. Since for all  $I \subseteq [t]$ , the map  $\varphi_I$  is

a homeomorphism, it is in particular bijective, so conditions of Corollary 3.12 are satisfied, and the defined map  $\varphi$  is a homeomorphism.  $\square$

**3.3. Main Theorem.** The fact that the protocol complexes in the immediate snapshot read/write shared memory model are homeomorphic to simplices has been folklore knowledge in the theoretical distributed computing community, [Her]. The next theorem provides a rigorous mathematical proof of this fact.

**Theorem 3.14.** *For every round counter  $\bar{r}$  there exists a homeomorphism*

$$\Phi(\bar{r}) : P(\bar{r}) \xrightarrow{\cong} P(\chi(\bar{r})),$$

such that

(1) *for all  $V \subset \text{supp } \bar{r}$  the following diagram commutes:*

$$(3.6) \quad \begin{array}{ccccc} P(\bar{r} \setminus V) & \xleftarrow{\delta_V(\bar{r})} & B_V(\bar{r}) & \xrightarrow{\beta_V(\bar{r})} & P(\bar{r}) \\ \Phi(\bar{r} \setminus V) \Big\downarrow \cong & & & & \Big\downarrow \cong \Phi(\bar{r}) \\ P(\chi(\bar{r} \setminus V)) & \xleftarrow{\delta_V(\chi(\bar{r}))} & B_V(\chi(\bar{r})) & \xrightarrow{\beta_V(\chi(\bar{r}))} & P(\chi(\bar{r})) \end{array}$$

(2) *for all  $S \subseteq \text{act } \bar{r}$  the following diagram commutes:*

$$(3.7) \quad \begin{array}{ccccccc} X_S(\bar{r}) & \xrightarrow{\gamma_S(\bar{r})} & P(\bar{r}_S) & \xrightarrow[\cong]{\Phi(\bar{r}_S)} & P(\chi(\bar{r}_S)) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_S) \xleftarrow{\gamma_S(\chi(\bar{r}))} X_S(\chi(\bar{r})) \\ & \searrow \alpha_S(\bar{r}) & & & & & \swarrow \alpha_S(\chi(\bar{r})) \\ & & P(\bar{r}) & \xrightarrow[\cong]{\Phi(\bar{r})} & P(\chi(\bar{r})) & & \end{array}$$

where  $\tau = \tau(\chi(\bar{r}_S), \chi(\bar{r})_S)$ .

In particular, the complex  $P(\bar{r})$  is simplicially homeomorphic to  $\Delta^{\text{supp } \bar{r}}$ .

**Proof.** Our proof is a double induction, first on  $|\text{supp } \bar{r}|$ , then, once  $|\text{supp } \bar{r}|$  is fixed, on the cardinality of the round counter  $\bar{r}$ . As a base of the induction, we note that the case  $|\text{supp } \bar{r}| = 1$  is trivial, since the involved spaces are points. Furthermore, if  $|\text{supp } \bar{r}|$  is fixed, and  $|\bar{r}| = 0$ , we take  $\Phi(\bar{r})$  to be the identity map. In this case the simplicial complexes  $P(\bar{r})$  and  $P(\chi(\bar{r}))$  are simplices. The diagram (3.6) commutes, since also  $\Phi(\bar{r} \setminus V)$  is the identity map. The condition (2) of the theorem is void, since  $\text{act } \bar{r} = \emptyset$ . As a matter of fact, more generally,  $\Phi(\bar{r})$  can be taken to be the identity map whenever  $\bar{r} = \chi(\bar{r})$ , that is whenever  $\bar{r}(i) \in \{0, 1\}$ , for all  $i \in \text{supp } \bar{r}$ .

We now proceed to prove the induction step, assuming that  $|\bar{r}| \geq 1$ . For every pair of sets  $A \subseteq S \subseteq \text{act } \bar{r}$ , such that  $S \neq \emptyset$ , we define a map

$$\varphi_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \longrightarrow X_{S,A}(\chi(\bar{r})),$$

as follows

$$(3.8) \quad \varphi_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \xrightarrow{\gamma_{S,A}(\bar{r})} P(\bar{r}_{S,A}) \xrightarrow[\cong]{\Phi(\bar{r}_{S,A})} P(\chi(\bar{r}_{S,A})) \xrightarrow[\cong]{\tau} P(\chi(\bar{r})_{S,A}) \xleftarrow{\gamma_{S,A}(\chi(\bar{r}))} X_{S,A}(\chi(\bar{r})),$$

where  $\tau = \tau(\chi(\bar{r}_{S,A}), \chi(\bar{r})_{S,A})$ . Since  $|\bar{r}_{S,A}| \leq |\bar{r}_S| = |\bar{r}| - |S| < |\bar{r}|$ , the map  $\Phi(\bar{r}_{S,A})$  is already defined by induction, so  $\varphi_{S,A}(\bar{r})$  is well-defined by the sequence (3.8). Obviously, the map  $\varphi_{S,A}$  is a homeomorphism for all pairs  $S, A$ .

We want to use Corollary 3.13 to construct the global homeomorphism  $\Phi(\bar{r})$  by gluing the local ones  $\varphi_{S,A}(\bar{r})$ . In our setting here, the notations of Corollary 3.13 translate to  $X = P(\bar{r})$ ,  $Y = P(\chi(\bar{r}))$ ,  $A_I$ 's are  $X_{S,A}(\bar{r})$ 's,  $B_I$ 's are  $X_{S,A}(\chi(\bar{r}))$ 's, and  $\varphi_I$ 's are  $\varphi_{S,A}(\bar{r})$ 's.

To satisfy the conditions of Corollary 3.13, we need to check that the following diagram commutes whenever  $X_{S,A} \subseteq X_{T,B}$

$$(3.9) \quad \begin{array}{ccc} X_{S,A}(\bar{r}) & \xrightarrow[\cong]{\varphi_{S,A}(\bar{r})} & X_{S,A}(\chi(\bar{r})) \\ \downarrow i & & \downarrow j \\ X_{T,B}(\bar{r}) & \xrightarrow[\cong]{\varphi_{T,B}(\bar{r})} & X_{T,B}(\chi(\bar{r})), \end{array}$$

where  $i$  and  $j$  denote the inclusion maps.

Note, that by Proposition 2.7, we have  $X_{S,A} \subseteq X_{T,B}$  if and only if either  $S = T$  and  $B \subseteq A$ , or  $T \subseteq A$ . Consider first the case  $S = T$ ,  $B = \emptyset$ . Consider the diagram on Figure 3.1. The leftmost pentagon is the diagram (2.9), which commutes by Proposition 2.16. The

$$\begin{array}{ccccccc} & & P(\bar{r}_{S,A}) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_{S,A})) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_{S,A}) \\ & \nearrow \gamma & \uparrow \delta & & \uparrow \delta & & \nwarrow \gamma \\ X_{S,A}(\bar{r}) & & B_A(\bar{r}_S) & & B_A(\chi(\bar{r}_S)) & & B_A(\chi(\bar{r})_S) & & X_{S,A}(\chi(\bar{r})) \\ \downarrow & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \downarrow \\ X_S(\bar{r}) & \xrightarrow[\gamma]{} & P(\bar{r}_S) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_S)) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_S) & \xleftarrow[\gamma]{} & X_S(\chi(\bar{r})) \end{array}$$

FIGURE 3.1. Commuting diagram used in the proof of Theorem 3.14

following hexagon is the diagram (3.6), where  $\bar{r}$  is replaced with  $\bar{r}_S$ . Since  $|\bar{r}_S| = |\bar{r}| - |S| < |\bar{r}|$ , this diagram commutes by induction. The next hexagon is the diagram (2.12), for  $\chi_{C_1,D_1} = \chi(\bar{r}_S)$ ,  $\chi_{C_2,D_2} = \chi(\bar{r})_S$ , and we use the fact that  $\chi(\bar{r}_S) \setminus A = \chi(\bar{r}_{S,A})$ . Finally, the rightmost pentagon is also the commuting diagram (2.9), where  $\bar{r}$  is replaced with  $\chi(\bar{r})$ . Since removing the 3 inner terms of the diagram on Figure 3.1 yields the diagram (3.9) with  $S = T$ ,  $B = \emptyset$ , we conclude that (3.9) commutes in this special case.

Consider now the case  $S = T$ ,  $B \subseteq A$ . We have inclusions  $X_{S,A} \hookrightarrow X_{S,B} \hookrightarrow X_S$ , and it is easy to see that the commutativity of the diagram (3.9) for the inclusion  $X_{S,A} \hookrightarrow X_{S,B}$  follows from the commutativity of the diagrams (3.9) for the inclusions  $X_{S,A} \hookrightarrow X_S$  and  $X_{S,B} \hookrightarrow X_S$ . Hence we are done with the proof of this case.

Let us now prove the commutativity of the diagram (3.9) for the inclusion  $X_{S,A} \hookrightarrow X_{T,B}$ , when  $T \subseteq A$ . Assume first that  $A = T = B \neq \emptyset$ , and consider the diagram on Figure 3.2, where  $\bar{S} = S \setminus A$ .

$$\begin{array}{ccccccc} & & P(\bar{r}_{S,A}) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_{S,A})) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_{S,A}) \\ & \nearrow \gamma & \uparrow \gamma & & \uparrow \gamma & & \nwarrow \gamma \\ X_{S,A}(\bar{r}) & & X_{\bar{S}}(\bar{r} \setminus A) & & X_{\bar{S}}(\chi(\bar{r} \setminus A)) & & X_{S,A}(\chi(\bar{r})) \\ \downarrow & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \\ X_{A,A}(\bar{r}) & \xrightarrow[\gamma]{} & P(\bar{r} \setminus A) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r} \setminus A)) & \xleftarrow[\gamma]{} & X_{A,A}(\chi(\bar{r})), \end{array}$$

FIGURE 3.2. Commuting diagram used in the proof of Theorem 3.14

A few of the maps in the diagram on Figure 3.2 need to be articulated. To start with, we have the identity  $\bar{r}_{A,A} = \bar{r} \setminus A$ , explaining the simplicial isomorphism  $\gamma_{A,A}(\bar{r}) : X_{A,A}(\bar{r}) \rightarrow P(\bar{r} \setminus A)$ . Similarly,  $\chi(\bar{r})_{A,A} = \chi(\bar{r} \setminus A)$  explains  $\gamma_{A,A}(\chi(\bar{r})) : X_{A,A}(\chi(\bar{r})) \rightarrow P(\chi(\bar{r} \setminus A))$ . Furthermore, by (1.2) we have  $\bar{r} \setminus A \downarrow \bar{S} = \bar{r} \downarrow S \setminus A = \bar{r}_{S,A}$  and  $\chi(\bar{r} \setminus A) \downarrow \bar{S} = \chi(\bar{r}) \setminus A \downarrow \bar{S} = \chi(\bar{r})_{S,A}$ . These identities explain the presence of the maps  $\gamma_{\bar{S}}(\bar{r} \setminus A) : X_{\bar{S}}(\bar{r} \setminus A) \rightarrow P(\bar{r}_{S,A})$ , and  $\gamma_{\bar{S}}(\chi(\bar{r} \setminus A)) : X_{\bar{S}}(\chi(\bar{r} \setminus A)) \rightarrow P(\chi(\bar{r})_{S,A})$ .

Let us look at the commutativity of the diagram on Figure 3.2. The middle heptagon is the diagram (3.7) with  $\bar{r} \setminus A$  instead of  $\bar{r}$  and  $\bar{S}$  instead of  $S$ ; where we again use the identity  $\bar{r} \setminus A \downarrow \bar{S} = \bar{r} \downarrow S \setminus A$ . Since  $|\text{supp}(\bar{r} \setminus A)| = |\text{supp} \bar{r}| - |A| < |\text{supp} \bar{r}|$ , the induction hypothesis implies that this heptagon commutes. The leftmost pentagon is (2.1), with  $\bar{S}$  instead of  $S$ , whereas the rightmost pentagon is (2.1) as well, this time with  $\bar{S}$  instead of  $S$ , and  $\chi(\bar{r})$  instead of  $\bar{r}$ . They both commute by Proposition 2.5. Again, removing the 2 inner terms from the diagram on Figure 3.2 will yield the diagram (3.9) with  $A = T = B$ , so we conclude that (3.9) commutes in this special case.

In general, when  $T \subseteq A$ , we have a sequence of inclusions  $X_{S,A} \hookrightarrow X_{S,T} \hookrightarrow X_{T,T} \hookrightarrow X_{T,B}$ . Again, it is easy to see that the commutativity of the diagram (3.9) for the inclusion  $X_{S,A} \hookrightarrow X_{T,B}$  follows from the commutativity of the diagrams (3.9) for the inclusions  $X_{S,A} \hookrightarrow X_{S,T}$ ,  $X_{S,T} \hookrightarrow X_{T,T}$ , and  $X_{T,T} \hookrightarrow X_{T,B}$ . Hence we are done with the proof of this case as well.

We now know that  $\Phi(\bar{r})$  is a well-defined homeomorphism between  $P(\bar{r})$  and  $P(\chi(\bar{r}))$ . To finish the proof of the main theorem, we need to check the commutativity of the diagrams (3.6) and (3.7). The commutativity of (3.7) is an immediate consequence of (3.8), and the way  $\Phi(\bar{r})$  was defined. To show that (3.6) commutes, pick any  $S \subseteq \text{act } \bar{r}$ , which is disjoint from  $A$ , and consider the diagram on Figure 3.3. The maps  $\varphi$  and  $\psi$  are as in

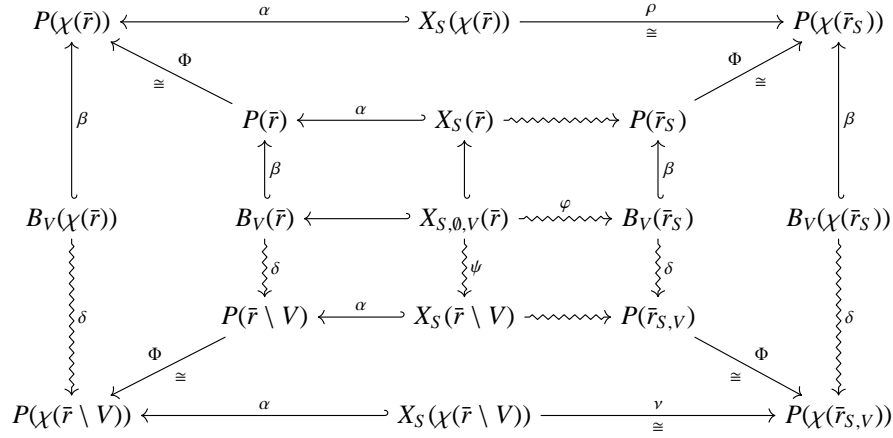


FIGURE 3.3. Commuting diagram used in the proof of Theorem 3.14

Proposition 2.15, and the maps  $\rho$  and  $\nu$  are given by

$$\rho : X_S(\chi(\bar{r})) \xrightarrow{\gamma_S(\chi(\bar{r}))} P(\chi(\bar{r}_S)) \xrightarrow{\tau(\chi(\bar{r}_S), \chi(\bar{r})_S)^{-1}} P(\chi(\bar{r}_S))$$

and

$$\nu : X_S(\chi(\bar{r} \setminus A)) \xrightarrow{\gamma_S(\chi(\bar{r} \setminus A))} P(\chi(\bar{r})_{S,A}) \xrightarrow{\tau(\chi(\bar{r}_{S,A}), \chi(\bar{r})_{S,A})^{-1}} P(\chi(\bar{r}_{S,A})),$$

$$\begin{array}{ccccc}
P(\chi(\bar{r})) & \xleftarrow{\alpha} & X_S(\chi(\bar{r})) & \rightsquigarrow^{\gamma} & P(\chi(\bar{r})_S) & \xleftarrow[\cong]{\tau} & P(\chi(\bar{r}_S)) \\
\uparrow \beta & & & & \uparrow \beta & & \uparrow \beta \\
B_V(\chi(\bar{r})) & & & & B_V(\chi(\bar{r})_S) & & B_V(\chi(\bar{r}_S)) \\
\downarrow \delta & & & & \downarrow \delta & & \downarrow \delta \\
P(\chi(\bar{r} \setminus V)) & \xleftarrow{\alpha} & X_S(\chi(\bar{r} \setminus V)) & \rightsquigarrow^{\gamma} & P(\chi(\bar{r})_{S,V}) & \xleftarrow[\cong]{\tau} & P(\chi(\bar{r}_S)_V)
\end{array}$$

FIGURE 3.4. Commuting diagram used in the proof of Theorem 3.14

where we use the identities  $\chi(\bar{r} \setminus A) = \chi(\bar{r}) \setminus A$ ,  $\chi(\bar{r}_S) \setminus A = \chi(\bar{r}_{S,A})$ , and  $\chi(\bar{r} \setminus A)_S = \chi(\bar{r})_{S,A}$ , with the latter one relying on the fact that  $S \cup A = \emptyset$ .

Let us investigate the diagram on Figure 3.3 in some detail. The middle part is precisely the diagram (2.8), which commutes by Proposition 2.15. We have 4 hexagons surround that middle part. The hexagon on the left is the diagram (3.6) itself. The hexagon above is precisely the diagram (3.7), so it commutes. The hexagon below is the diagram (3.7) with  $\bar{r} \setminus A$  instead of  $\bar{r}$ , where we use (1.3) again. This diagram commutes by the induction hypothesis. The hexagon on the right is the diagram (3.6) with  $\bar{r}_S$  instead of  $\bar{r}$ . Since  $|\bar{r}_S| < |\bar{r}|$ , it also commutes by the induction assumption.

Let us now show that the diagram obtained from the one on Figure 3.3 by the removal of the 9 inner terms commutes. This diagram can be factorized as shown on Figure 3.4. The left part of the diagram on Figure 3.4 is (2.8) with  $\chi(\bar{r})$  instead of  $\bar{r}$ , whereas the right part of the diagram on Figure 3.4 is the diagram (2.12) with  $\chi_{C_1,D_1} = \chi(\bar{r}_S)$ ,  $\chi_{C_2,D_2} = \chi(\bar{r})_S$ . They both commute, hence so does the whole diagram.

Consider now two sequences of maps in the diagram on Figure 3.3:

$$(3.10) \quad B_V(\bar{r}) \cap X_S(\bar{r}) \hookrightarrow B_V(\bar{r}) \xrightarrow{\beta} P(\bar{r}) \xrightarrow[\cong]{\Phi} P(\chi(\bar{r}))$$

and

$$(3.11) \quad X_{S,\emptyset,V}(\bar{r}) \hookrightarrow B_V(\bar{r}) \rightsquigarrow^{\delta} P(\bar{r} \setminus V) \xrightarrow[\cong]{\Phi} P(\chi(\bar{r} \setminus V)) \xleftarrow[\delta]{\sim} B_V(\chi(\bar{r})) \xrightarrow{\beta} P(\chi(\bar{r}))$$

It follows by a simple diagram chase that the commutativities in the diagram on Figure 3.3 which we have shown imply the equality of these two maps. This is true for all  $S$ , such that  $S \subseteq \text{act } \bar{r}$  and  $S \cap V = \emptyset$ . On the other hand, the subcomplexes  $X_{S,\emptyset,V}(\bar{r})$ , where  $S \subseteq \text{act } \bar{r}$ ,  $S \cap V = \emptyset$ , cover  $B_V(\bar{r})$ . As a matter of fact, the simplicial isomorphisms  $\psi$  and  $\delta_V(\bar{r})$  show that they induce a stratification which is isomorphic to the stratification of  $P(\bar{r} \setminus V)$  by  $X_S(\bar{r} \setminus V)$ . The fact that they cover  $B_V(\bar{r})$  completely implies that the maps (3.10) and (3.11) remain the same after the first term is skipped, which is the same as to say that (3.6) commutes. This concludes the proof.  $\square$

**Corollary 3.15.** *For an arbitrary round counter  $\bar{r}$  the immediate snapshot complex  $P(\bar{r})$  is homeomorphic to the closed ball of dimension  $|\text{supp } \bar{r}| - 1$ .*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BREMEN, 28334 BREMEN, FEDERAL REPUBLIC OF GERMANY  
*E-mail address:* dfk@math.uni-bremen.de